AN INVERSE VARIABLE TECHNIQUE IN THE MHD-EQUILIBRIUM PROBLEM

L.M. DEGTYAREV and V.V. DROZDOV

Keldysh Institute of Applied Mathematics, USSR Academy of sciences, Moscow, USSR

1985
NORTH-HOLLAND – AMSTERDAM

Contents

Computer Physics Reports 2 (1985) 341–387
North-Holland, Amsterdam

AN INVERSE VARIABLE TECHNIQUE IN THE MHD-EQUILIBRIUM PROBLEM

L.M. DEGTYAREV and V.V. DROZDOV

Keldysh Institute of Applied Mathematics, USSR Academy of Sciences, Moscow, USSR

Received 8 November 1984; in revised form 10 January 1985

The inverse variable method for finding two-dimensional MHD equilibrium configurations is discussed in detail. The computational technique is based on the formulation of the equilibrium problem in flux coordinates, which is equivalent to a first-order system of equations. The method allows one to obtain coordinates of the magnetic surfaces directly from the system of equations. Examples of some equilibrium problems illustrating the possibilities of the inverse variable method are given.

1. Introduction

The plasma equilibrium problem has been discussed extensively in connection with controlled thermonuclear fusion in toroidal magnetic systems. Solutions of this problem allow one to obtain feasible plasma configurations and external electrical currents needed to provide equilibrium control. The exact description of equilibrium configurations is necessary to study their MHD stability as well as transport processes for particles, energy and magnetic fields.

In axisymmetric systems the search for equilibrium is reduced to the solution of the two-dimensional elliptic Grad–Shafranov equilibrium equation with a quasilinear right-hand side [1,2] for the so-called flux function $\psi(r, z)$. The isolines $\psi(r, z) = \text{const}$ yield cross sections of the magnetic surfaces in a plane (r, z). The mathematical problems arising here have some specific features. In their various statements, a free boundary and an infinite region may occur. In the problems of physical interest, the equilibrium equation, due to its right-hand side, becomes a nonlinear and integro-differential one [3,4]. One needs to know the geometry of the magnetic surfaces with high accuracy for a subsequent analysis of the equilibrium configuration stability and simulation of transport processes.

The analytic investigations of plasma equilibrium in magnetic fields yield only qualitative results [2]. Extensive and thorough analysis of possible equilibria, and the optimization of specific installations require the use of computerized techniques. The traditional and most versatile numerical techniques are based on approximation of an equilibrium equation of a fixed grid in the plane (r, z) [5,6]. Such algorithms involve a tiresome procedure of constructing the mapping lines $\psi(r, z) = \text{const}$. Therefore they are not optimal from the viewpoint of using the results for the stability and evolution problems.

Recently, an approach has proved useful for constructing equilibrium computation algorithms allowing one to obtain magnetic surfaces (being nested) in an explicit form. It is based on introduction of natural (flux) coordinates (a, Θ) [2,7,8] such that the basic variable a ("radius") is constant on the isolines $\psi(r, z) = \text{const}$, and the auxiliary variable Θ ("angle") is constant on some curves coming out from the extremum points of $\psi(r, z)$ – the magnetic axis. There is a wide arbitrariness in a choice of a and especially Θ. With the aid of the replacement $(r, z) \to (a, \Theta)$, the original equilibrium problem may be formulated straightforwardly for the coordinates of the magnetic surfaces $r(a, \Theta)$ and $z(a, \Theta)$. Such a replacement will henceforth be called the variable inversion. Three numerical techniques are used now to implement the above approach. In the first one $r(a, \Theta)$, $z(a, \Theta)$ are searched for as a series in a chosen system of functions with coefficients depending only on a. A set of ordinary differential equations for the coefficients is obtained from the variational statement of the equilibrium problem [2,9–11]. Their complex nonlinear structure restricts the number of series terms. The second technique consists in multiple computations of the elliptic equilibrium problem on successive grids that are constructed on isolines of $\psi(r, z)$ and irregular in (r, z) [12,13]. This technique goes along with the traditional ones and requires no less computational times. Finally, the third method, the inverse variable technique, is based on a direct (a, Θ)-approximation of the first-order nonlinear system of equations equivalent to the original equilibrium problem [14–19]. It has successfully been used for solving the applied MHD equilibrium problems [20–27].

This paper deals with a detailed discussion of the inverse variable technique and its application to elliptic plane problems which include, in particular, the search for symmetric equilibrium plasma configurations.

In section 2 typical formulations of equilibrium problems are considered for axisymmetric systems. The inverse variable procedure for the Dirichlet problem using quasipolar ($\Theta(r, z) =$ const are straight lines) and quasiorthogonal ($a(r, z) =$ const, $\Theta(r, z) =$ const are orthogonal) coordinates are described in section 3. The quasipolar coordinates for a numerical method based on the variational formulation of the 3-D MHD problem were earlier used in ref. [28]. The equilibrium problem in flux variables is formulated in section 4, and the respective computation algorithm is discussed in sections 5 through 7. In section 8 some exact solutions of the Grad–Shafranov equation are given that are later used as tests. In section 9 the testing results are presented to describe the accuracy of the quasipolar coordinate inversion. Also, computational examples for various equilibrium problems are given to illustrate the algorithm performance.

2. Axisymmetric MHD-equilibrium problems

2.1. Equilibrium equation

The plasma equilibrium is described by the one-fluid ideal MHD equations

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B}, \quad \boldsymbol{j} = \nabla \times \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{2.1}$$

where p is the gasdynamic plasma pressure, $\boldsymbol{B}$ is the magnetic field induction, $\boldsymbol{j}$ is the electric current density. In the axisymmetric case the system (2.1) may be considerably simplified yielding the well-known Grad–Shafranov equation [1] for the flux function

$$\frac{1}{r}\Delta^*\psi = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right) = -j_\varphi(r, \psi). \tag{2.2}$$

Here (r, φ, z) are cylindrical coordinates, $j_\varphi(r, \psi)$ is the current density projection onto the toroidal direction. For equilibrium configurations it follows from (2.1) that the toroidal surfaces $\psi =$ const coincide with the magnetic, isobaric ($p(\psi) = p$) and flux ($f(\psi) = f$) surfaces. In this case

$$\begin{aligned} &\boldsymbol{B} = \nabla\psi \times \nabla\varphi + f\nabla\varphi, \\ &\boldsymbol{j} = \nabla f \times \nabla\varphi + rj_\varphi\nabla\varphi, \\ &j_\varphi = r\frac{\mathrm{d}p}{\mathrm{d}\psi} + \frac{1}{2r}\frac{\mathrm{d}f^2}{\mathrm{d}\psi}, \\ &\psi = (1/2\pi)\Psi, \quad f = (1/2\pi)F, \end{aligned} \tag{2.3}$$

where Ψ and F are the poloidal field and current fluxes with respect to the symmetry axis $r = 0$.

Thus the axisymmetric plasma MHD-equilibrium is determined by the solution of the quasilinear elliptic eqs. (2.2), (2.3). Now let us consider the ways of giving the functions $p(\psi)$, $f(\psi)$ and boundary value conditions.

2.2. Equilibrium problem with given $p(\psi)$, $f(\psi)$

Let the plasma (Ω_p, $\Gamma_p = \partial\Omega_p$) be maintained by an ideally conducting wall (Γ_0) separated

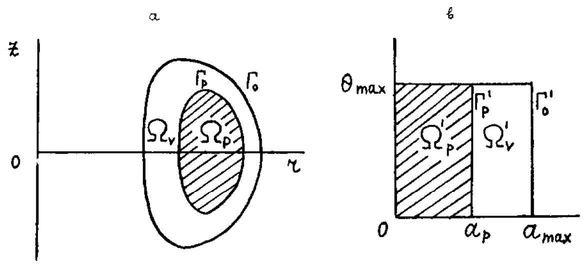

Fig. 1.

from it by a vacuum region Ω_v (fig. 1a). Then, by giving some $p(\psi)$ and $f(\psi)$, we have due to (2.3)

$$j_\varphi(r, \psi) = \begin{cases} r\dfrac{dp}{d\psi} + \dfrac{1}{r}f\dfrac{df}{d\psi}, & (r, z) \in \Omega_p \\ 0, & (r, z) \in \Omega_v. \end{cases}$$

The wall Γ_0 and the unknown plasma boundary Γ_p are obviously magnetic surfaces so that

$$\begin{aligned} &\psi(r, z) = \psi_0 = 0, \quad (r, z) \in \Gamma_0, \\ &\psi(r, z) = \psi_p, \quad (r, z) \in \Gamma_p. \end{aligned} \tag{2.4}$$

The value of ψ_p is defined by giving the limiter $(r_p, z_p) \in \Gamma_p$

$$\psi_p = \psi(r_p, z_p) \tag{2.5}$$

or the relation

$$\psi_p/\psi_{max} = \alpha < 1, \tag{2.5'}$$

where ψ_{max} is the value of $\psi(r, z)$ on the magnetic axis (r^m, z^m) with $\nabla\psi = 0$. If the surface current is absent at the plasma boundary the solution is continuous together with the normal derivative

$$[\partial\psi/\partial n] = 0, \quad (r, z) \in \Gamma_p. \tag{2.6}$$

2.3. *Equilibrium problem with given* $p(\psi)$, $q(\psi)$

When studying the quasiequilibrium evolution of the toroidal plasma another formulation of the equilibrium problem is widely used, the mathematical nature of which considerable differs from the above one (2.2)–(2.6). For example, during the evolution of an ideally conducting plasma with the given time dependent pressure function $p(\psi, t)$ the magnetic fluxes are conserved due to the frozen-in field condition. Hence, the conservation condition extends to the poloidal flux

$$\psi_{max} - \psi_p = \bar{\psi}_{max} \tag{2.7}$$

and the distribution of the safety factor over the magnetic surfaces

$$q(\psi) = -\mathrm{d}\Phi/\mathrm{d}\Psi,$$

where Φ is the toroidal flux through the cross section $\psi = \text{const}$. According to the definition of $f(\psi)$, $q(\psi)$ we have

$$f(\psi) = 4\pi^2 q(\psi) \Big/ \oint_{\psi=\text{const}} \frac{\mathrm{d}s}{r^2 |\nabla\psi|}, \tag{2.8}$$

the integration being over the area of the toroidal surface $\psi = \text{const}$.

Thus the considered evolution requires the solution of the equilibrium problem (2.2)–(2.8) at each instant for the known functions $p(\psi, t)$ and $q(\psi)$. Let us consider some specific features of this problem. By introducing the averaging over the volume $V(\psi)$ between "adjacent" magnetic surfaces

$$\langle g \rangle = \frac{\mathrm{d}}{\mathrm{d}V} \int_V g \,\mathrm{d}\tau = \oint_{\psi=\text{const}} g \frac{\mathrm{d}s}{|\nabla\psi|} \Big/ \oint_{\psi=\text{const}} \frac{\mathrm{d}s}{|\nabla\psi|},$$

we obtain

$$f(\psi) = -4\pi^2 q(\psi) \left\langle \frac{1}{r^2} \right\rangle^{-1} \frac{\mathrm{d}\psi}{\mathrm{d}V},$$

and (2.2) in the form of the nonlinear integro-differential equation

$$\begin{aligned} &\nabla \cdot \left(\frac{1}{r^2} \nabla\psi \right) + \frac{1}{r^2} c \frac{\mathrm{d}}{\mathrm{d}V} c \frac{\mathrm{d}\psi}{\mathrm{d}V} = -\frac{\mathrm{d}p}{\mathrm{d}\psi}, \\ &c = 4\pi^2 q(\psi) \left\langle \frac{1}{r^2} \right\rangle^{-1}, \quad (r, z) \in \Omega_\mathrm{p}. \end{aligned} \tag{2.9}$$

Averaging (2.9) yields

$$\frac{\mathrm{d}}{\mathrm{d}V} \left(\left\langle \frac{|\nabla V|^2}{r^2} \right\rangle \frac{\mathrm{d}\psi}{\mathrm{d}V} \right) + 16\pi^4 q \frac{\mathrm{d}}{\mathrm{d}V} \left(q \left\langle \frac{1}{r^2} \right\rangle^{-1} \frac{\mathrm{d}\psi}{\mathrm{d}V} \right) = -\frac{\mathrm{d}p}{\mathrm{d}\psi}.$$

From this it is seen that in order to obtain a unique solution of eq. (2.9) one should give $\bar{\psi}_{\max}$ in the condition (2.7) or some other value, e.g.

$$\mathrm{d}\psi/\mathrm{d}V = -\lambda, \quad V = 0, \tag{2.7'}$$

corresponding to the toroidal field given on the magnetic axis. Note that it is the second term in the operator of (2.9) that is dominant. This must be taken into account when constructing the algorithms for numerical solution of the above equilibrium problem.

2.4. Adiabatic equilibrium

Consider the adiabatic evolution of the equilibrium plasma configuration. Let $N(\psi)$ be the number of plasma particles in the volume $V(\psi)$. Then the density is given by

$$n(\psi)=\frac{\mathrm{d}N}{\mathrm{d}V}=\frac{\mathrm{d}N}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}V}$$

and, hence, due to adiabaticity with the exponent γ, the pressure is given by

$$p(\psi)=\eta(\psi)\left(-\frac{\mathrm{d}\psi}{\mathrm{d}V}\right)^{\gamma},\quad \eta(\psi)=\frac{p_0(\psi)}{n_0^{\gamma}(\psi)}\left(-\frac{\mathrm{d}N}{\mathrm{d}\psi}\right)^{\gamma}. \tag{2.10}$$

As a result the adiabatic equilibrium problem is reduced to the solution of (2.2)–(2.7), (2.10) for the given distributions $f(\psi)$ and $\eta(\psi)$. Like in the case of the known $p(\psi)$, $q(\psi)$ it is quite obvious that the condition (2.7) must be given if (2.2) is rewritten in the form of an integro-differential equation (called the generalized one in ref. [4])

$$\nabla\cdot\left(\frac{1}{r^2}\nabla\psi\right)+\frac{\mathrm{d}}{\mathrm{d}\psi}\eta\left(-\frac{\mathrm{d}\psi}{\mathrm{d}V}\right)^{\gamma}=-\frac{1}{2r^2}\frac{\mathrm{d}f^2}{\mathrm{d}\psi},\quad (r,\ z)\in\Omega_{\mathrm{p}}.$$

2.5. *Equilibrium in an external field*

From the practical viewpoint, the problem of the plasma equilibrium being maintained by the given field of external circular currents $J_1, J_2, \ldots, J_K$ with the coordinates $(R_1, Z_1), (R_2, Z_2), \ldots, (R_K, Z_K)$ is of most interest. In this case it is convenient to search for $\psi(r, z)$ in the form

$$\psi(r,\ z)=\psi_{\mathrm{i}}(r,\ z)+\psi_{\mathrm{e}}(r,\ z), \tag{2.11}$$

where $\psi_{\mathrm{i}}(r, z)$ corresponds to the plasma currents (the inherent plasma field) and $\psi_{\mathrm{e}}(r, z)$ to the currents in external coils.

The Green's function for (2.2) in an infinite region is

$$G(r,\ z;\ r',\ z')=\frac{\sqrt{rr'}}{\pi t}\left[\left(1-\frac{t^2}{2}\right)K(t)-E(t)\right], \tag{2.12}$$

$$t^2=\frac{4rr'}{(r+r')^2+(z-z')^2},$$

where $K(t)$, $E(t)$ are the first and second kind elliptic integrals, respectively. Then

$$\psi_{\mathrm{e}}(r,\ z)=\sum_{j=1}^{K}J_jG(r,\ z;\ R_j,\ Z_j), \tag{2.13}$$

and the function $\psi_i(r, z)$ satisfies the equation

$$\frac{1}{r}\Delta^*\psi_i = \begin{cases} -j_\varphi(r, \psi_i + \psi_e), & (r, z) \in \Omega_p, \\ 0, & (r, z) \overline{\in} \Omega_p. \end{cases} \tag{2.14}$$

Here the region occupied by the plasma Ω_p is determined by the inequality $\psi(r, z) > \psi_p$. The boundary conditions (2.4)–(2.6) should be supplemented by the condition that the solution vanishes at infinity and on the torus axis

$$\psi_i(r, z) \to 0$$

$$r^2 + z^2 \to \infty, \quad r \to 0. \tag{2.15}$$

Thus the external field equilibrium problem under the given $p(\psi)$ and $f(\psi)$ is reduced to the solution of (2.14), (2.11)–(2.13), (2.4)–(2.6), (2.15). In the case of the known $p(\psi)$ and $q(\psi)(f(\psi), \eta(\psi))$ one should additionally give condition (2.7).

Now we give an auxiliary expression for the flux function $\psi_i(r, z)$ of the inherent plasma field, and we shall use it when constructing the computational algorithm. By applying the generalized Green's formula to the function $\bar{\psi}(r, z) = \psi(r, z) - \psi_p$ the volume integral

$$\psi_i(r, z) = \iint_{\Omega_p} G(r, z; r', z') j_\varphi(r', \psi) \mathrm{d}r' \mathrm{d}z' \tag{2.16}$$

may be reduced to the contour one

$$\psi_i(r, z) = -\int_{\Gamma_p} \frac{1}{r'} \frac{\partial \psi}{\partial n} G(r, z; r', z') \mathrm{d}s'$$

$$+\eta(r, z)(\psi(r, z) - \psi_p),$$

$$\eta(r, z) = \begin{cases} 1, & (r, z) \in \Omega_p \\ 0.5, & (r, z) \in \Gamma_p \\ 0, & (r, z) \overline{\in} \Omega_p \cup \Gamma_p, \end{cases} \tag{2.17}$$

where n is the external normal to the boundary Γ_p.

2.6. Calculation of the external confinement field

Together with the above direct problem of equilibrium in an external field, where $\psi(r, z)$ is determined for the given currents J_l, an inverse problem is also of interest. The latter consists in searching for the given currents J_l (and generally their positions) that would provide an equilibrium configuration with the given form of cross section and distributions $p(\psi)$, $f(\psi)$ or

$p(\psi)$ and $q(\psi)$. Inverse problems of such kind may be treated in terms of optimal control described by a nonlinear elliptic equation [29].

Now, it is required to find the control $J=(J_1, J_2, \ldots, J_K) \in E_K$ provided that the plasma occupies the region close to Ω, whose boundary Γ is called the control contour. Formally it means

$$\psi(r, z)=\psi_p, \quad (r, z) \in \Gamma.$$

Let $\psi(r, z; J)$ be the solution to the direct problem of equilibrium in an external field, $(r_s z_s)$, $s=1, 2, \ldots, m$ are the observation points on Γ. Then the discussed inverse problem may be considered in the variational statement

$$W(J)=\sum_{s=1}^{m} \sigma_s\left[\psi(r_s, z_s; J)-\psi_p\right]^2+\alpha\left[\sum_{l=1}^{K} \delta_l J_l^2+\delta_0 \psi_p^2\right], \tag{2.18}$$

$$W(J_0)=\min W(J), \quad J \in E_\partial,$$

where E_∂ is a set of admissible controls σ_s, δ_l and δ_0 are weight functions. From the terminological viewpoint [29] (2.18) is the problem of the one-point control with boundary observation.

Note that the optimal control problem (2.18) is ill posed, and therefore the need for introducing a stabilizing functional is obvious enough. The multiplier α plays here the role of the regularization parameter [30]. The value ψ_p may also be varied and found from the minimum condition for $W(J, \psi_p)$.

3. Inverse variables in elliptic plane problems

3.1. Inverse variables, a governing equation

In this section a version of the inverse variable technique is proposed for application to elliptic plane problems. The search for equilibrium in symmetric plasma configurations is reduced to the solution of these problems.

Let Ω be a confined region in the plane with boundary $\partial\Omega$. Consider the Dirichlet problem with a homogeneous boundary condition for the quasilinear elliptic equation

$$\mathscr{L}u=\sum_{k,l=1}^{2} \frac{\partial}{\partial x_k}\left(a_{kl}(x, u)\frac{\partial u}{\partial x_l}\right)=-f(x, u), \tag{3.1}$$

$$\sum_{k,l=1}^{2} a_{kl}\xi_k\xi_l \geqslant c^2\left(\xi_1^2+\xi_2^2\right), \quad x \in \Omega$$

$$u(x)=u^0, \quad x \in \partial\Omega. \tag{3.2}$$

Let the solution of (3.1), (3.2): (1) exist and be unique; (2) have an unique extremum u^m at an internal point $x^m \in \Omega$ so that $u(x)$ possesses nested contours. Without discussion of requirements on the input data of the problem (3.1), (3.2) we note that it allows introducing an analog of the polar coordinates connected to the solution.

We introduce new independent variables by performing the smooth nonsingular transformation $(x_1, x_2) \to (\xi_1, \xi_2)$. Then eq. (3.1) turns into

$$\mathscr{L}u = J \sum_{k,l=1}^{2} \frac{\partial}{\partial \xi_k}\left(h_{kl} J \frac{\partial u}{\partial \xi_l} \right) = -f, \tag{3.3}$$

$$\begin{aligned}
h_{11} &= a_{11}\left(\frac{\partial x_2}{\partial \xi_2}\right)^2 - (a_{12}+a_{21})\left(\frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_2}\right) + a_{22}\left(\frac{\partial x_1}{\partial \xi_2}\right)^2,\\
h_{12} &= -a_{11}\left(\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}\right) + a_{12}\left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}\right) + a_{21}\left(\frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1}\right) - a_{22}\left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}\right),\\
h_{21} &= -a_{11}\left(\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}\right) + a_{12}\left(\frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1}\right) + a_{21}\left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2}\right) - a_{22}\left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2}\right),\\
h_{22} &= a_{11}\left(\frac{\partial x_2}{\partial \xi_1}\right)^2 - (a_{12}+a_{21})\left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_1}\right) + a_{22}\left(\frac{\partial x_1}{\partial \xi_1}\right)^2,
\end{aligned} \tag{3.4}$$

$$J = \frac{\partial(\xi_1, \xi_2)}{\partial(x_1, x_2)} = \left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} \right)^{-1}.$$

Let $(\xi_1, \xi_2) = (a, \Theta)$ be an analog of the polar coordinates with an origin at a point $z^m \in \Omega$ so that $a \in [0, \alpha_{max})$ is "a radius", and $\Theta \in [0, \Theta_{max})$ is "an angle". Then besides the boundary condition (3.2) and the Θ-variable periodicity conditions one should require for eq. (3.3) the regularity of solution for $a = 0$:

$$J\left(h_{11}\frac{\partial u}{\partial a} + h_{12}\frac{\partial u}{\partial \Theta} \right) = 0, \quad a = 0. \tag{3.5}$$

Let the variable a uniquely depend on the solution of the original problem

$$a(x^m) = 0, \quad a(x) = a_{max}, \quad x \in \partial\Omega,$$

$$u(a) = u^0 + k\int_a^{a_{max}} \nu(a')\mathrm{d}a'. \tag{3.6}$$

Here $\nu(a) > 0$ is the given function, and k is an unknown constant,

$$k = (u^m - u^0) \Big/ \int_0^{a_{max}} \nu(a')\mathrm{d}a'. \tag{3.7}$$

The choice of $\nu(a)$ is determined by the problem's nature. In the simplest case $\nu(a)=1/a_{\max}$, then

$$u(a)=(u^{\mathrm{m}}-u^0)(1-a/a_{\max})+u^0.$$

We give the function $\Theta(x)$ at the boundary of the region $x=x^0(\Theta)$, $x\in\partial\Omega$ and choose it to be monotonically increasing from 0 to $\Theta_{\max}$ in the counterclockwise direction. Moreover, an additional restriction must be imposed on the function $\Theta(x)$ to determine its unique choice. These restrictions specify a version of the inverse variable technique. Note that in the replacement $(x_1, x_2)\to(a, \Theta)$ the original region Ω is mapped into the given rectangle $\Omega'=[0, a_{\max})\times[0, \Theta_{\max})$ and the Jacobian $J>0$.

Let us formulate the problem (3.1), (3.2) in variables by assuming the functions $x_i(a, \Theta)$ and the value k to be unknown. Consider the auxiliary function

$$I(a, \Theta)=-\frac{\partial(u, \Theta)}{\partial(x_1, x_2)}=-\frac{\mathrm{d}u}{\mathrm{d}a}J=k\nu(a)J(a, \Theta).$$

Due to (3.6), $I(a, \Theta)$ satisfies the equation

$$\frac{\partial}{\partial a}(h_{11}I)+\frac{\partial}{\partial\Theta}(h_{21}I)=fJ^{-1}, \tag{3.8}$$

with the boundary conditions

$$h_{11}I=0, \quad a=0, \tag{3.9}$$

$$I(a, \Theta+\Theta_{\max})=I(a, \Theta). \tag{3.10}$$

For $x_i(a, \Theta)$ by taking into account the definition $I(a, \Theta)$ we obtain the equation

$$\frac{\partial(x_1, x_2)}{\partial(a, \Theta)}=k\nu I^{-1} \tag{3.11}$$

which must be supplemented by the condition for choosing $\Theta(x)$. An obvious consequence of (3.11) is

$$k=S(a)\left[\int_0^a\int_0^{\Theta_{\max}}\frac{\nu(a')}{I(a', \Theta')}\mathrm{d}a'\mathrm{d}\Theta'\right]^{-1}, \tag{3.12}$$

where $S(a)$ is the area of the region $\Omega_a\in\Omega$, bounded by isoline $a(x)=\mathrm{const}$.

3.2. Quasipolar coordinates

Let us call the inverse coordinates the quasipolar ones if $\Theta(x)=\mathrm{const}$ on the rays outcoming from the extremum point x^{m}. Due to this

$$x_i(a, \Theta)=x_i^{\mathrm{m}}+\rho(a, \Theta)\left[x_i^0(\Theta)-x_i^{\mathrm{m}}\right]. \tag{3.13}$$

For $0 \leqslant \rho(a, \Theta) \leqslant 1$ the function $\rho(a, \Theta)$ is monotonic in a, for which

$$\rho(0, \Theta) = 0, \quad \rho(a_{\max}, \Theta) = 1. \tag{3.14}$$

Note, that introduction of quasipolar coordinates is possible only when any region $\Omega_a \in \Omega$ is starwise with respect to the extremum point x^{m}. By substituting (3.13) into (3.11) we obtain the equation for $\rho(a, \Theta)$

$$\frac{\partial \rho^2}{\partial a} = \frac{2k}{\Delta} \frac{\nu}{I}, \tag{3.15}$$

$$\Delta(x^{\mathrm{m}}, x^0) = (x_1^0 - x_1^{\mathrm{m}}) \frac{\partial x_2^0}{\partial \Theta} - (x_2^0 - x_2^{\mathrm{m}}) \frac{\partial x_1^0}{\partial \Theta}.$$

From the two conditions (3.14) we shall keep the first one as a boundary condition for (3.15), namely

$$\rho(0, \Theta) = 0 \tag{3.16}$$

and the second one is used to eliminate the constant:

$$k = 0.5\Delta(x^{\mathrm{m}}, x^0) \left[\int_0^{a_{\max}} \frac{\nu}{I} \mathrm{d}a \right]^{-1} = \text{const}. \tag{3.17}$$

Finally we obtain

$$\frac{\partial \rho^2}{\partial a} = \frac{\nu}{I} \left[\int_0^{a_{\max}} \frac{\nu}{I} \mathrm{d}a \right]^{-1}. \tag{3.18}$$

Thus the problem (3.1), (3.2) is reduced to a nonlinear system of two first-order equations: eq. (3.8) with the conditions (3.9), (3.10) and eq. (3.18) with the conditions (3.16), (3.17). The solution determines the point x^{m}, the function $\rho(a, \Theta)$, and hence, the coordinate lines $x_i(a, \Theta)$.

Note that the following generalization is possible. The above approach may be used for the Dirichlet problem in the doubly connected regions Ω. Let $u(x)$ satisfy (3.1) in Ω and the boundary conditions

$$u(x) = u^0, \quad x \in \partial\Omega^0,$$

$$u(x) = u^{\mathrm{m}} > u^0, \quad x \in \partial\Omega^{\mathrm{m}}, \tag{3.19}$$

where $\partial\Omega^0$, $\partial\Omega^{\mathrm{m}}$ are the external and internal boundaries of Ω. We introduce the variables (a, Θ) so that the isolines $\Theta(x) = \text{const}$ should be nonintersecting straight lines that connect the points $x^0(\Theta)$ and $x^{\mathrm{m}}(\Theta)$ on the contours $\partial\Omega^0$, $\partial\Omega^{\mathrm{m}}$. As before, the function $\rho(a, \Theta)$ is determined by the equality (3.13). For it the equations and conditions (3.15)–(3.18) are valid with

$$I \to \tilde{I} = (\tilde{J}/J)I,$$

$$\Delta \to \tilde{\Delta} = \left(x_1^0 - x_1^{\mathrm{m}}\right)\frac{\partial\left(x_2^0 - x_2^{\mathrm{m}}\right)}{\partial\Theta} - \left(x_2^0 - x_2^{\mathrm{m}}\right)\frac{\partial\left(x_1^0 - x_1^{\mathrm{m}}\right)}{\partial\Theta}, \tag{3.20}$$

$$\tilde{J} = \frac{\partial\left(x_1 - x_1^{\mathrm{m}},\ x_2 - x_2^{\mathrm{m}}\right)}{\partial(a,\ \Theta)}.$$

The regularity condition (3.9) for (3.8) is replaced here by

$$I(a,\ \Theta) = k\nu(0)J(0,\ \Theta). \tag{3.21}$$

The value k in the problem (3.1), (3.19) is known and given by the equality (3.7).

3.3. Quasiorthogonal coordinates

We shall call the inverse coordinates the quasiorthogonal ones if the function $\Theta(x)$ satisfies the equation

$$\gamma^2\frac{\partial u}{\partial x_1}\frac{\partial\Theta}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial\Theta}{\partial x_2} = 0, \tag{3.22}$$

where $\gamma(x) \neq 0$ is a certain given function. Specifically, at $\gamma(x) = 1$ the isolines $a(x) = \text{const}$ and $\Theta(x) = \text{const}$ are orthogonal. Note that if the solution $u(x)$ is known then due to (3.2) the coordinate lines $\Theta(x) = \text{const}$ are determined by the equation

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{1}{\gamma^2}\frac{\partial u}{\partial x_2}\left[\frac{\partial u}{\partial x_1}\right]^{-1},$$

with the condition $x = x^0(\Theta)$ at the boundary $\partial\Omega$. Eq. (3.21) in the inverse coordinates takes the form

$$\frac{1}{\gamma}\frac{\partial x_1}{\partial a}\frac{\partial x_1}{\partial\Theta} + \gamma\frac{\partial x_2}{\partial a}\frac{\partial x_2}{\partial\Theta} = 0,$$

From this it follows for $x_i(a,\ \Theta),\ i = 1,\ 2$

$$\mu\frac{\partial x_1}{\partial a} = \gamma\frac{\partial x_2}{\partial\Theta},\quad \mu\frac{\partial x_2}{\partial a} = -\frac{1}{\gamma}\frac{\partial x_1}{\partial\Theta}, \tag{3.23}$$

where $\mu(a,\ \Theta)$ with involvement of (3.11) is connected with $I(a,\ \Theta)$ by the equality

$$\mu = \frac{I}{k\nu}\left[\frac{1}{\gamma}\left(\frac{\partial x_1}{\partial\Theta}\right)^2 + \gamma\left(\frac{\partial x_2}{\partial\Theta}\right)^2\right] \tag{3.24}$$

or

$$\mu^{-1} = \frac{I}{k\nu}\left[\frac{1}{\gamma}\left(\frac{\partial x_1}{\partial a}\right)^2 + \gamma\left(\frac{\partial x_2}{\partial a}\right)^2\right]. \tag{3.24'}$$

The boundary conditions for (3.23) are coordinates of the boundary $\partial\Omega$ and the Θ-periodicity conditions

$$\begin{aligned} &x_i(a_{\max}, \Theta) = x_i^0(\Theta), \\ &x_i(a, \Theta + \Theta_{\max}) = x_i(a, \Theta), \quad i = 1, 2. \end{aligned} \tag{3.25}$$

Besides, at the extremum point $x(0, \Theta) = x^{\mathrm{m}}$ eqs. (3.23) degenerate and reduce to the regularity conditions

$$\mu \partial x_i/\partial a = 0, \quad a = 0, \quad i = 1, 2. \tag{3.26}$$

Thus, like the case of quasipolar coordinates the original problem (3.1), (3.2) is reformulated in the form of a nonlinear system of first-order eqs. (3.8)–(3.10), (3.23)–(3.26).

For a numerical solution it is expedient to transfer from the Cauchy–Riemann system (3.23) to the two elliptic equations

$$\begin{aligned} \mathscr{L}_1 x_1 &= \frac{\partial}{\partial a}\left(\frac{\mu}{\gamma}\frac{\partial x_1}{\partial a}\right) + \frac{\partial}{\partial \Theta}\left(\frac{1}{\mu\gamma}\frac{\partial x_1}{\partial \Theta}\right) = 0, \\ \mathscr{L}_2 x_2 &= \frac{\partial}{\partial a}\left(\mu\gamma\frac{\partial x_2}{\partial a}\right) + \frac{\partial}{\partial \Theta}\left(\frac{\gamma}{\mu}\frac{\partial x_2}{\partial \Theta}\right) = 0. \end{aligned} \tag{3.27}$$

In order for this approach be adequate it is necessary to conserve eq. (3.23) as the boundary condition for (3.27) at fixed $a = a_*$.

In closing this section we shall make two remarks. First, a choice of the function $u(a)$ may be implicit. The instead of (3.6) the function $k\nu(a) = -\mathrm{d}u/\mathrm{d}a$ is determined from an additional condition. For example, if a coincides with the area $S(u)$ inside the isoline $u(x) = \text{const}$ then

$$\frac{\mathrm{d}S}{\mathrm{d}a} \equiv 1 = -\frac{\mathrm{d}u}{\mathrm{d}a}\int_0^{\Theta_{\max}} \frac{1}{I}\,\mathrm{d}\Theta.$$

Second, sometimes it is expedient to transform the initial system of coordinates (x_1, x_2) into the new one (y_1, y_2) and then to inverse the variables $(y_1, y_2) \to (a, \Theta)$. The level lines $\Theta(x) = \text{const}$ will have the form of the given curves outcoming from the point $x^{\mathrm{m}} = x(y^{\mathrm{m}})$.

4. Equilibrium equation in flux coordinates

For the equilibrium problem (2.2), (2.3), by going over to the functions

$$\psi \to R\psi, \quad f \to Rf,$$

where R is the characteristic torus radius, we have in terms of (3.1), (3.2)

$$(x_1, x_2) = (r, z), \quad u = \psi,$$

$$a_{11} = a_{22} = R/r, \quad a_{12} = a_{21} = 0,$$

$$f(x, u) = j_\varphi(r, \psi) = \begin{cases} \dfrac{r}{R}\dfrac{\mathrm{d}p}{\mathrm{d}\psi} + \dfrac{R}{2r}\dfrac{\mathrm{d}f^2}{\mathrm{d}\psi}, & \psi_p < \psi < \psi_{max} \\ 0, & \psi_0 < \psi < \psi_p. \end{cases} \tag{4.1}$$

For given $p(\psi)$ and $q(\psi)$, owing to (2.8), we have

$$f(a) = \frac{2\pi}{\Theta_{max}} q(a) \left\langle \frac{1}{rI} \right\rangle^{-1},$$

$$\langle g \rangle = \frac{1}{\Theta_{max}} \int_0^{\Theta_{max}} g(a, \Theta)\mathrm{d}\Theta. \tag{4.2}$$

Now, we introduce the factor characterizing the equilibrium problem

$$\kappa = \begin{cases} 1, & p(\psi),\ f(\psi) \text{ are given;} \\ 0, & p(\psi),\ q(\psi) \text{ are given.} \end{cases} \tag{4.3}$$

By taking into account (4.1)–(4.3) eq. (3.8) for the function

$$I = -\partial(\psi, \Theta)/\partial(r, z)$$

will have the form

$$\begin{aligned} &\frac{\partial}{\partial a}(h_{11} I) + \frac{\partial}{\partial \Theta}(h_{21} I) \\ &\quad + (1-\kappa)\left(\frac{2\pi}{\Theta_{max}}\right)^2 \frac{R}{r} J^{-1} q \left\langle \frac{1}{rJ} \right\rangle^{-1} \frac{\mathrm{d}}{\mathrm{d}a}\left(q \left\langle \frac{1}{rI} \right\rangle^{-1} \right) \\ &= \left(-\frac{r}{R}\frac{\mathrm{d}p}{\mathrm{d}\psi} + \kappa \frac{R}{2r}\frac{\mathrm{d}f^2}{\mathrm{d}\psi} \right) \frac{1}{J}, \quad (a, \Theta) \in \Omega'_p, \\ &\frac{\partial}{\partial a} h_{11} I + \frac{\partial}{\partial \Theta}(h_{21} I) = 0, \quad (a, \Theta) \in \Omega'_v, \end{aligned} \tag{4.4}$$

where $\Omega'_p = [0, a_p) \times [0, \Theta_{max})$, $\Omega'_v = (a_p, a_{max}) \times [0, \Theta_{max})$ are the mappings of the regions occupied by the plasma Ω_p and the vacuum layer Ω_v (fig. 1b). At the plasma–vacuum boundary Γ_p, which is mapped into the straight line $a = a_p$, the condition for continuity of the normal derivative of solution (2.6) turns into

$$[I]_{a_p} = I(a_p + 0, \Theta) - I(a_p - 0, \Theta) = 0. \tag{4.5}$$

The value a_p is determined by the conditions (2.5) or (2.5′) and (3.6). If on the edge magnetic surface of the plasma there is a surface current, then (4.5) should be replaced by the total pressure balance condition

$$[p + \tfrac{1}{2}B^2]_{\Gamma_p} = 0.$$

In this case since

$$I = -\frac{\partial\psi}{\partial n}\frac{\partial\Theta}{\partial\tau}, \quad \boldsymbol{B} = R(\nabla\psi \times \nabla\varphi + f\nabla\varphi)$$

we obtain

$$[I^2]_{a_p} = -\left(2\frac{r^2}{R^2}[p]_{a_p} + [f^2]_{a_p}\right)\left(\frac{\partial\Theta}{\partial\tau}\right)^2.$$

Here the derivative along the magnetic surface

$$\frac{\partial\Theta}{\partial\tau} = \left[\left(\frac{\partial r}{\partial\Theta}\right)^2 + \left(\frac{\partial z}{\partial\Theta}\right)^2\right]^{-1/2}$$

and the coordinate $r(a, \Theta)$ are continuous at the plasma boundary Γ_p; the discontinuities $[p]_{a_p}$, $[f^2]_{a_p}$ are assumed to be given. As it has been noted in section 2.3, for eq. (4.4) (considering $x_i(a, \Theta)$ to be given) at $\kappa = 0$ a condition on the magnetic axis $a = 0$ is required in addition to (3.9), (3.10). Such a condition can be obtained from (2.7) which according to (3.6), (3.7) is equivalent to giving the value k. Therefore by taking into account the definition of $I(a, \Theta)$ we have

$$\left\langle \frac{1}{rI} \right\rangle^{-1} = k\nu \left\langle \frac{1}{rJ} \right\rangle^{-1}, \quad a = 0. \tag{4.6}$$

5. Iterational process of the inverse variable algorithm

The formulation of the problem (3.1), (3.2) in the form of a nonlinear system of two first-order equations raises a question on the method of its solution. In this section, remaining within the framework of a differential problem we shall discuss a natural two-stage iteration process. At the first stage by starting from the approximation $x_i^n(a, \Theta)$ the function $I^{n+1}(a, \Theta)$ is obtained from eq. (3.8) for fixed coefficients and the right hand side. The second stage consists in determining a new approximation $x_i^{n+1}(a, \Theta)$ from respective equations for the obtained $I^{n+1}(a, \Theta)$. Here we shall restrict ourselves to the case of inverse quasipolar coordinates which has been developed to date in detail. As to the orthogonal coordinates, the details concerning one of the first algorithms for the implementation of the inverse variable technique may be found in ref. [17]. In this case the iterational process considerably complicates at the first stage due to the structure of equations for the functions $x_i(a, \Theta)$.

5.1. Successive approximations in doubly-connected region case

We first discuss the iterational process for the Dirichlet problem in the doubly-connected region Ω. Let $x_i^n(a, \Theta)$ be known on the iteration with superscript n, i.e. a family of coordinate lines $a^n(x) = \text{const}$, $\Theta^n(x) = \text{const}$ is given in Ω. We shall search in the rectangle $\Omega' = [0, a_{\max}) \times [0, \Theta_{\max})$ for a solution of the problem corresponding to (3.8), (3.10), (3.21) that is given by

$$L^n w^{n+1} - D^n w^{n+1} = f^n,$$

$$w^{n+1}(0, \Theta) = k\nu(0),$$

$$w^{n+1}(a, \Theta + \Theta_{\max}) = w^{n+1}(a, \Theta), \tag{5.1}$$

where

$$w^{n+1} = I^{n+1}/J^n,$$

$$J^n = \left[\frac{\partial(x_1^n, x_2^n)}{\partial(a, \Theta)}\right]^{-1}, \quad f^n = f(x^n(a, \Theta), u^n(a)),$$

$$Lw = J\left[\frac{\partial}{\partial a}(h_{11}Jw) + \frac{\partial}{\partial \Theta}(h_{21}Jw)\right]. \tag{5.2}$$

The summand Dw in (5.1) has the sense of a regularizator (the artificial viscosity in a difference problem) and may be chosen in the form

$$Dw = J\frac{\partial^2}{\partial a \partial \Theta}\left(\alpha J\frac{\partial w}{\partial \Theta}\right),$$

$$\alpha = \alpha_0 d(x^0, x^{\mathrm{m}}) d(x, x^{\mathrm{m}}), \tag{5.3}$$

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

$$\alpha_0 = \text{const}.$$

Since the problem is considered in the doubly-connected region the value k is known and given by equality (3.7).

Thus, from (5.1)–(5.3) we obtain $I^{n+1} = J^n w^{n+1}$. We note that the function $w^{n+1}(a, \Theta)$ depends, in general, on the angular variable Θ. The iterational process is aimed at such redetermination of coordinate lines $x_i^n(a, \Theta)$ that would provide the limiting equality

$$w^n(a, \Theta) \underset{n\to\infty}{\to} w(a) = k\nu(a). \tag{5.4}$$

Indeed, at the limit (5.4) for the function

$$u(a) = u^0 + \int_a^{a_{\max}} w(a')\mathrm{d}a'$$

we have

$$Lw = -\mathscr{L}u,$$

$$Dw = -J\frac{\partial^2}{\partial a \partial \Theta}\left(\alpha J \frac{\partial^2 u}{\partial a \partial \Theta}\right) = 0$$

and, hence,

$$\mathscr{L}u = -f,$$

i.e. $u = u(a(x))$ is the solution of the problem (3.1), (3.19).

Now we shall go over to the second stage of iterational process – the determination of $\rho(a, \Theta)$. By substituting $I^{n+1}(a, \Theta)$ into (3.18), (3.20) we obtain

$$\frac{\partial(\rho^{n+1})^2}{\partial a} = \frac{\nu}{\tilde{I}^{n+1}}\left[\int_0^{a_{\max}} \frac{\nu}{\tilde{I}^{n+1}}\mathrm{d}a\right]^{-1},$$

$$\tilde{I}^{n+1} = \frac{\tilde{J}^n}{J^n}I^{n+1} = \frac{2w^{n+1}}{\tilde{\Delta}\partial(\rho^n)^2/\partial a},$$

or as the final result with involvement of the boundary condition (3.16):

$$\frac{\partial(\rho^{n+1})^2}{\partial a} = \mu^{n+1}\frac{\nu}{w^{n+1}}\frac{\partial(\rho^n)^2}{\partial a}, \quad \rho^{n+1}(0, \Theta) = 0,$$

$$\mu^{n+1}(\Theta) = \left[\int_0^{a_{\max}} \frac{\nu(a)}{w^{n+1}(a, \Theta)}\frac{\partial(\rho^n)^2}{\partial a}\mathrm{d}a\right]^{-1}. \tag{5.5}$$

From (5.5) we obtain $\rho^{n+1}(a, \Theta)$ and then, according to (3.13), the new approximation $x_i^{n+1}(a, \Theta)$.

Let us show that the limiting equality (5.4) is satisfied at the convergence of the iterational process, so it means that the latter provides the solution of the problem posed. Really, if iterations converge, then from (5.5) for any $a \in [0, a_{\max})$, including $a = 0$, it follows

$$\mu^{n+1}(\Theta)\frac{\nu(a)}{w^{n+1}(a, \Theta)} \underset{n\to\infty}{\to} 1.$$

From this, by taking into account the boundary condition (5.1), we have

$$\mu^{n+1}(\Theta)\frac{1}{k} \underset{n\to\infty}{\to} 1.$$

From the last two equalities we obtain (5.4).

Note that for the problem in the doubly-connected region, when transferring to a next iteration only the isolins $a^n(x) = \text{const}$ are reconstructed while the rays $\Theta(x) = \text{const}$ are conserved.

5.2. Iterations in the singly-connected region case

We generalize the above iterational technique for the problem (3.1), (3.2) in the singly-connected region Ω when coordinates of the extremum x^{m} and the solution u^{m} at it are unknown. By assuming that $w^{n+1}(a, \Theta) = w_\epsilon^{n+1}(a)$ for $0 < a < \epsilon$ eq. (5.1) is averaged in Θ here. We obtain the problem

$$\frac{\mathrm{d}}{\mathrm{d}a}\langle h_{11}^n J^n\rangle w_\epsilon^{n+1} = \langle f^n/J^n\rangle, \quad 0 < a < \epsilon$$

$$\langle h_{11}^n J^n\rangle w_\epsilon^{n+1} = 0, \quad a = 0, \tag{5.6}$$

where

$$\langle \varphi\rangle = \frac{1}{\Theta_{\max}}\int_0^{\Theta_{\max}} \varphi(a, \Theta)\mathrm{d}\Theta.$$

The solution (5.6) for $\epsilon < a < a_{\max}$ and the solution (5.1) with the boundary condition $w^{n+1}(\epsilon, \Theta) = w_\epsilon^{n+1}(\epsilon)$ yields

$$w^{n+1}(a, \Theta) = \begin{cases} w_\epsilon^{n+1}(a), & 0 < a < \epsilon, \\ w^{n+1}(a, \Theta), & \epsilon \leqslant a \leqslant a_{\max}. \end{cases}$$

As above we then determine $\rho^{n+1}(a, \Theta)$ and $x_i^{n+1}(a, \theta)$ where the coordinates of the extremum $(x^{\mathrm{m}})^{n+1}$ may be found from the equality

$$(x_i^{\mathrm{m}})^{n+1} = \frac{\langle x_i^{n+1}(J^{n+1/2})^{-1}\rangle}{\langle (J^{n+1/2})^{-1}\rangle}, \quad a = \epsilon. \tag{5.7}$$

Here

$$J^{n+1/2}=\left[\tfrac{1}{2}\Delta\left[(x^{m})^{n},\, x^{0}\right]\frac{\partial(\rho^{n+1})^{2}}{\partial a}\right]^{-1}.$$

Finally, k^{n+1} is obtained from formula (3.12) as

$$k^{n+1}=S(a_{\max})\left[\int_0^{a_{\max}}\int_0^{\Theta_{\max}}\frac{\nu}{I^{n+1}}\mathrm{d}a\mathrm{d}\Theta\right]^{-1}, \tag{5.8}$$

so that

$$u^{n+1}(a)=u^{0}+k^{n+1}\int_a^{a_{\max}}\nu(a')\mathrm{d}a'.$$

If this iterational process converges at the fixed ϵ then the limiting relation (5.4) is satisfied as before. By tending ϵ to zero we obtain in the limit the solution $u(x)$ of the problem (3.1), (3.2) as coordinates of its level lines.

6. Difference approximation

In the rectangle Ω' we introduce the grid $\bar{\omega}'_h=$

$$\left\{a_i=a_{i-1}+h^1_{i-1/2},\, 1\leqslant i\leqslant N,\, a_0=0,\, a_N=a_{\max};\right.$$
$$\left.\Theta_j=\Theta_{j-1}+h^2_{j-1/2},\, 1\leqslant j\leqslant M,\, \Theta_0=0,\, \Theta_M=\Theta_{\max}\right\}.$$

In the plane x the grid $\bar{\omega}'_h$ corresponds to an irregular quadrangular grid $\bar{\omega}_h=\{x_{ij}=x(a_i,\, \Theta_j)\}$. On $\bar{\omega}_h$ we consider the grid problem

$$\begin{aligned}\mathscr{L}_h u_h &= -f_h, && x\in\omega_h\\ u_h &= u^0, && x\in\gamma_h\end{aligned} \tag{6.1}$$

that approximates (3.1), (3.2).

A discrete analog of the variable inversion described in section 3 is as follows. It is required to find the grid $\bar{\omega}_h$ (topologically equivalent to a radial circular one) such that the solution of problem (6.1) on it should not depend on the index j and must satisfy the condition

$$u_h\equiv u_i=u^0+k\sum_{l=i+1}^{N}\nu_{l-1/2}h^1_{l-1/2}. \tag{6.2}$$

The approximation of a nonlinear system of eqs. (3.8)–(3.10), (3.16)–(3.18) is constructed in such a way that it is algebraically equivalent to the problem (6.1), (6.2). The solution of the difference scheme obtained may be found by using the iteration technique from section 5.

6.1. Variational-difference scheme in the Dirichlet problem

The scheme (6.1) is constructed by applying the variational principle [31] as follows:

$$\Phi(v)=\iint_{\Omega}\left(\sum_{k,l=1}^{2} a_{kl}\frac{\partial v}{\partial x_k}\frac{\partial v}{\partial x_l}-2vf\right)\mathrm{d}x_1\mathrm{d}x_2,$$

$$\min \Phi(v)=\Phi(u),\quad v\in \mathring{W}_2^1(\Omega). \tag{6.3}$$

Using the coordinates $\xi_1=a$, $\xi_2=\Theta$ the functional (6.3) is

$$\Phi(v)=\iint_{\Omega'}\left(\sum_{k,l=1}^{2} h_{kl}J^2\frac{\partial v}{\partial \xi_l}\frac{\partial v}{\partial \xi_k}-2vf\right)J^{-1}\mathrm{d}\xi_1\mathrm{d}\xi_2. \tag{6.4}$$

By turning to the grid functions v_h we approximate (6.4) as

$$\Phi_h(v_h)=\sum_{i,j}\left\{\left(\sum_{k,l}[h_{kl}J^2]_h\left[\frac{\partial v}{\partial \xi_l}\frac{\partial v}{\partial \xi_k}\right]_h-2[vf]_h\right)\frac{h^1h^2}{[J]_h}\right\}_{i-1/2,j-1/2}. \tag{6.5}$$

Here $[\varphi]_{h,i-1/2,j-1/2}$ is the approximation of $\varphi(a,\Theta)$ at the center of a rectangular cell $\Omega'_{i-1/2,j-1/2}$ with apexes $(i,\ j;\ i-1,\ j;\ i-1,\ j-1;\ i,\ j-1)$. Now we divide $\Omega'_{i-1/2,j-1/2}$ into four triangles pairwise and give the apexes at direct angles the index $S=1,\ 2,\ 3,\ 4$, respectively. With regard to (3.4), we assume

$$[pq]_h=0.25\sum_s p_h^s q_h^s,$$

$$\left[\frac{\partial p}{\partial \xi_l}\frac{\partial q}{\partial \xi_k}\right]_h=0.25\sum_s\left(\frac{\delta p}{\delta \xi_p}\right)^s\left(\frac{\delta q}{\delta \xi_k}\right)^s, \tag{6.6}$$

$$[J]_h=J\left\{\left[\frac{\partial x_m}{\partial \xi_l}\frac{\partial x_n}{\partial \xi_k}\right]_h\right\},\quad [h_{kl}J^2]_h=[h_{kl}]_h[J]_h^2,$$

$$[h_{kl}]_h=h_{kl}\left\{\left[\frac{\partial x_m}{\partial \xi_l}\frac{\partial x_n}{\partial \xi_k}\right]_h,\ [\dot{a}_{kl}]_h\right\}.$$

The difference variables $(\delta\varphi/\delta\xi)^s$ are determined in the usual way; for example, for $s=1$

$$\left(\frac{\delta\varphi}{\delta a}\right)^1=\left(\frac{\delta\varphi}{\delta a}\right)_{i-1/2,j}=\frac{\varphi_{ij}-\varphi_{i-1,j}}{h^1_{i-1/2}},\quad \left(\frac{\delta\varphi}{\delta\Theta}\right)^1=\left(\frac{\delta\varphi}{\delta\Theta}\right)_{i,j-1/2}=\frac{\varphi_{ij}-\varphi_{i,j-1}}{h^2_{j-1/2}}.$$

Note that by using the approximation technique (6.6), the summation in (6.5) depends only on coordinates of apexes of the curvilinear quadrangle $\Omega_{i-1/2,j-1/2}$ (geometry of the grid $\bar{\omega}_h$) and the grid function v_h, f_h at the apexes. The steps $h^1_{i-1/2}$, $h^2_{j-1/2}$ characterizing the grid $\bar{\omega}'_h$ in (6.5), (6.6) are practically absent, for the fixed $\bar{\omega}_h$ they may be given any values. Consequently, the approximation $\Phi_h(v_h)$ is consistent formally with both forms (6.3) and (6.4) of the initial functional $\Phi(v)$. The difference scheme (6.1) is obtained by minimizing the quadratic functional (6.5) [32]. It may be written in the form corresponding to eq. (3.3) as

$$\begin{aligned}
\mathscr{L}_h u_h = \left(\frac{J}{h^1h^2}\right)_{ij} \Big\{ & \tfrac{1}{2}\left[\left(h_{11}Jh^2\right)_{i+1/2,j+1/2} + \left(h_{11}Jh^2\right)_{i+1/2,j-1/2}\right] \frac{\delta u}{\delta a}_{i+1/2,j} \\
& - \tfrac{1}{2}\left[\left(h_{11}Jh^2\right)_{i-1/2,j+1/2} + \left(h_{11}Jh^2\right)_{i-1/2,j-1/2}\right] \frac{\delta u}{\delta a}_{i-1/2,j} \\
& + \tfrac{1}{4}\Big[\left(h_{21}Jh^1\right)_{i+1/2,j+1/2}\left(\frac{\delta u}{\delta a}_{i+1/2,j+1} + \frac{\delta u}{\delta a}_{i+1/2,j}\right) \\
& + \left(h_{21}Jh^1\right)_{i-1/2,j+1/2}\left(\frac{\delta u}{\delta a}_{i-1/2,j+1} + \frac{\delta u}{\delta a}_{i-1/2,j}\right) \\
& - \left(h_{21}Jh^1\right)_{i+1/2,j-1/2}\left(\frac{\delta u}{\delta a}_{i+1/2,j} + \frac{\delta u}{\delta a}_{i+1/2,j-1}\right) \\
& - \left(h_{21}Jh^1\right)_{i-1/2,j-1/2}\left(\frac{\delta u}{\delta a}_{i-1/2,j} + \frac{\delta u}{\delta a}_{i-1/2,j-1}\right)\Big] \\
& + \tfrac{1}{4}\Big[\left(h_{12}Jh^2\right)_{i+1/2,j+1/2}\left(\frac{\delta u}{\delta \Theta}_{i+1,j+1/2} + \frac{\delta u}{\delta \Theta}_{i,j+1/2}\right) \\
& + \left(h_{12}Jh^2\right)_{i+1/2,j-1/2}\left(\frac{\delta u}{\delta \Theta}_{i+1,j-1/2} + \frac{\delta u}{\delta \Theta}_{i,j-1/2}\right) \\
& - \left(h_{12}Jh^2\right)_{i-1/2,j+1/2}\left(\frac{\delta u}{\delta \Theta}_{i,j+1/2} + \frac{\delta u}{\delta \Theta}_{i-1,j+1/2}\right) \\
& - \left(h_{12}Jh^2\right)_{i-1/2,j-1/2}\left(\frac{\delta u}{\delta \Theta}_{i,j-1/2} + \frac{\delta u}{\delta \Theta}_{i-1,j-1/2}\right)\Big] \\
& + \tfrac{1}{2}\left[\left(h_{22}Jh^1\right)_{i+1/2,j+1/2} + \left(h_{22}Jh^1\right)_{i-1/2,j+1/2}\right] \frac{\delta u}{\delta \Theta}_{i,j+1/2} \\
& - \tfrac{1}{2}\left[\left(h_{22}Jh^1\right)_{i+1/2,j-1/2} + \left(h_{22}Jh^1\right)_{i-1/2,j-1/2}\right] \frac{\delta u}{\delta \Theta}_{i,j-1/2} \Big\}
\end{aligned}$$

$$= \left(L_h \frac{\delta u}{\delta a}\right)_{ij} + \left(K_h \frac{\delta u}{\delta \Theta}\right)_{ij} = -f_{ij}, \quad x_{ij} \in \omega_h, \tag{6.7}$$

$$(J/h^1h^2)_{ij} = \left[\tfrac{1}{4}\sum(h^1h^2/J)_{i\pm1/2,j\pm1/2}\right]^{-1},$$

$$u_{N_j} = u^0, \quad u_{i,j+M} = u_{ij}.$$

At the central node $x_{0j} = x^{\mathrm{m}}$ we have a nonlocal boundary condition

$$u_{0j} = u^{\mathrm{m}}, \quad f_{0j} = f^{\mathrm{m}},$$

$$\sum_{j=1}^{M}\left\{\left[(h_{11}Jh^2)_{1/2,j-1/2} + (h_{11}Jh^2)_{1/2,j+1/2}\right]\frac{\delta u}{\delta a}_{1/2,j}\right.$$

$$\left. + (h_{12}Jh^2)_{1/2,j-1/2}\frac{\delta u}{\delta \Theta}_{1,j-1/2} + f_{0j}\left(\frac{h^1h^2}{J}\right)_{1/2,j-1/2}\right\} = 0,$$

which approxmates the regularity condition (3.5).

6.2. Variable inversion in a grid problem

We use the approximation (3.8)–(3.10) or (5.1). The operator L_h is obtained from (6.7) by assuming

$$\frac{\delta u}{\delta \Theta} = 0, \quad -\frac{\delta u}{\delta a} = w_h = (I/J)_h.$$

For the regularizator D in (5.1) we use the natural approximation

$$D_h w_h = \left(\frac{J}{h^1h^2}\right)_{ij}\left\{\left[\left(\alpha J\frac{\delta w}{\delta \Theta}\right)_{i+1/2,j+1/2} - \left(\alpha J\frac{\delta w}{\delta \Theta}\right)_{i+1/2,j-1/2}\right]\right.$$

$$\left. - \left[\left(\alpha J\frac{\delta w}{\delta \Theta}\right)_{i-1/2,j+1/2} - \left(\alpha J\frac{\delta w}{\delta \Theta}\right)_{i-1/2,j-1/2}\right]\right\}.$$

As a result we obtain the difference analog (5.1)

$$L_h w_h - D_h w_h = f_h, \quad \xi \in \omega_h', \tag{6.8}$$

with the conditions

$$w_{1/2,j} \equiv w_{1/2}, \quad w_{i+1/2,j+M} = w_{i+1/2,j},$$

$$w_{1/2}\sum_{j=1}^{M}(h_{11}Jh^2)_{1/2,j-1/2} = \tfrac{1}{2}f^m\sum_{j=1}^{M}\left(\frac{h^1h^2}{J}\right)_{1/2,j-1/2}. \tag{6.9}$$

The last equality in (6.9) corresponds to the averaged one-dimensional eq. (5.6). The problem

(6.8), (6.9) is two-point in i and three-point in j. Its solution may be found with the aid of cyclic computational sweep.

6.3. Quasipolar coordinates

We apply the iterational technique from section 5. By knowing the approximation of grid ω_h^n we solve (6.8), (6.9) and obtain w_h^{n+1}. Then we determine the solution of the problem corresponding to (5.5):

$$\frac{\rho_{ij}^{n+1}-\rho_{i-1,j}^{n+1}}{h_{i-1/2}^1}=\mu_j^{n+1}\frac{\nu_{i-1/2}}{w_{i-1/2,j}^{n+1}}\frac{\rho_{ij}^n-\rho_{i-1,j}^n}{h_{i-1/2}^1}$$

$$\rho_{0,j}^{n+1}=0,\quad \mu_j^{n+1}=\left[\sum_{i=1}^{N}\frac{\nu_{i-1/2}}{w_{i-1/2,j}^{n+1}}\left(\rho_{ij}^n-\rho_{i-1,j}^n\right)\right]^{-1}.$$

By taking into account (5.7) we obtain the new approximation of coordinates of the extremum point

$$(x^{\mathrm{m}})^{n+1}=\frac{\sum_{j=1}^{M}\rho_{1j}^{n+1}x_j^0\left(\frac{h^2}{J^{n+1/2}}\right)_{1/2,j}}{\sum_{j=1}^{M}\rho_{1j}^{n+1}\left(\frac{h^2}{J^{n+1/2}}\right)_{1/2,j}},$$

$$\left(\frac{h^2}{J^{n+1/2}}\right)_{1/2,j}=\frac{(\Delta^n h^2)_j}{2}\frac{\left(\rho_{1j}^{n+1}\right)^2-\left(\rho_{0j}^{n+1}\right)^2}{h_{1/2}^1},$$

$$(\Delta^n h^2)_j=\tfrac{1}{2}\Big[\big((x_1^0)_j-(x_1^{\mathrm{m}})^n\big)\big((x_2^0)_{j+1}-(x_2^0)_{j-1}\big)$$
$$-\big((x_2^0)_j-(x_2^{\mathrm{m}})^n\big)\big((x_1^0)_{j+1}-(x_1^0)_{j-1}\big)\Big],$$

so that the nodes ω_h^{n+1} are obtained from (3.13). Finally, according to (5.8)

$$k^{n+1}=2S_h\left[\sum_{j=1}^{M}\frac{(\Delta^n h^2)_j}{\mu_j^{n+1}}\right]^{-1},\quad S_h=\tfrac{1}{2}\sum_{j=1}^{M}(\Delta^n h^2)_j,$$

where S_h is an area of the polygon Ω_h with apexes x_j^0.

Like in the differential case the convergence of iterations signifies the solution of the difference problem (6.1), (6.2).

6.4. Equilibrium equation approximation

Now we shall discuss in detail the approximation of the problem (4.1)–(4.6) corresponding to the axisymmetric equilibrium. In the case of the given distributions $p(\psi)$ and $q(\psi)$ the nonlinear integro-differential operator, featured in (4.4),

$$Fw=\frac{2\pi}{\Theta_{\max}}Rq\left\langle\frac{R}{rJ}\right\rangle^{-1}\frac{R}{r}\frac{\mathrm{d}f}{\mathrm{d}a},$$

$$f=\frac{2\pi}{\Theta_{\max}}Rq\left\langle\frac{R}{rJw}\right\rangle^{-1}$$

is approximated by

$$(F_hw_h)_{ij}=\left(\frac{c}{h^1}\right)_{ij}(f_{i+1/2}-f_{i-1/2}),$$

$$f_{i-1/2}=\begin{cases}\dfrac{2\pi}{\Theta_{\max}}Rq_{i-1/2}\left\langle\dfrac{R}{rJw}\right\rangle_{i-1/2}, & 1\leqslant i\leqslant N_\mathrm{p}\\ f_{N_\mathrm{p}-1/2}, & N_\mathrm{p}<i\leqslant N,\end{cases}\tag{6.10}$$

$$\left(\frac{c}{h^1}\right)_{ij}=\frac{2\pi}{\Theta_{\max}}R\left(\frac{J}{h^1h^2}\right)_{ij}\tfrac{1}{4}\sum q_{i\pm1/2}\left\langle\frac{R}{rJ}\right\rangle^{-1}_{i\pm1/2}\left(\frac{R}{r}\frac{h^2}{J}\right)_{i\pm1/2,j\pm1/2}.$$

Here N_p is the number of the isoline describing the plasma boundary $a_\mathrm{p}=a_{N_\mathrm{p}}$ and

$$\left\langle\frac{R}{rJ}\right\rangle_{i-1/2}=\frac{1}{\Theta_{\max}}\sum_{j=1}^{M}\left(\frac{R}{r}\frac{h^2}{J}\right)_{i-1/2,j-1/2},$$

$$\left\langle\frac{R}{rJw}\right\rangle_{i-1/2}=\frac{1}{2}\frac{1}{\Theta_{\max}}\sum_{j=1}^{M}\left(\frac{R}{r}\frac{h^2}{J}\right)_{i-1/2,j-1/2}\left(\frac{1}{w_{i-1/2,j}}+\frac{1}{w_{i-1/2,j-1}}\right).$$

The approximation of the right hand side is chosen in the form

$$(\varphi_h)_{ij}=\begin{cases}\left(\dfrac{J}{h^1h^2}\right)_{ij}\tfrac{1}{4}\sum\varphi_{i\pm1/2,j\pm1/2}\left(\dfrac{h^1h^2}{J}\right)_{i\pm1/2,j\pm1/2}, & 1\leqslant i\leqslant N_\mathrm{p}\\ 0, & N_\mathrm{p}\leqslant i<N.\end{cases}\tag{6.11}$$

As a result we have the scheme

$$L_hw_h+(1-\kappa)F_hw_h-D_hw_h=\varphi_h,\quad \xi\in\omega_h',$$

$$w_{i+1/2,j+M}=w_{i+1/2,j},\quad w_{1/2,j}\equiv w_{1/2},\tag{6.12}$$

$$w_{1/2}\langle h_{11}J\rangle_{1/2}+(1-\kappa)\frac{2\pi}{\Theta_{\max}}Rq_{1/4}(f_{1/2}-f_0)=0.5h^1_{1/2}\left\langle\frac{\varphi}{J}\right\rangle_{1/2},$$

$$f_{1/2}=\frac{2\pi}{\Theta_{\max}}Rq_{1/2}w_{1/2}\left\langle\frac{R}{rJ}\right\rangle^{-1}_{1/2},\quad w_{1/2}=kv_{1/2}. \tag{6.13}$$

In the condition (6.13) the quantity k (and, hence, $w_{1/2}$) is known for $\kappa=0$ and, according to (2.5), (2.7), (3.6), determined by the equality

$$k=\bar{\psi}_{\max}\Big/\sum_{i=1}^{N_p}v_{i-1/2}h^1_{i-1/2}. \tag{6.14}$$

Therefore, the relation (6.13) may be used for obtaining f_0 which, due to (2.7′) and (2.8), is equal to

$$f_0=4\pi^2q_0(r^{\mathrm{m}})^2\lambda. \tag{6.14′}$$

On the contrary, if in the equilibrium problem the condition (2.7′) is given, eqs. (6.13) and (6.14′) allow one to find $w_{1/2}$ and, thus, the quantity k.

Thus, the scheme (6.10)–(6.14) is a difference analog of (4.1)–(4.6). Now we show a way of finding the solution to (6.12) on the nth iteration step, i.e. for the fixed coefficients and right hand side. As it has already been noted for (6.8), (6.9), for each $1\leqslant i\leqslant N-1$ the scheme reduces to the j-(three-point) problem with a nonlinear integral term when $\kappa=0$:

$$A_jY_{j+1}-C_jY_j+B_jY_{j-1}-(1-\kappa)Q_jX=\Phi_j,\quad 0\leqslant j\leqslant M-1,$$

$$X=\sum_{j=1}^{M}\alpha_j(Y)Y_j,$$

$$Y_{j+M}=Y_j,\quad j=-1,0. \tag{6.15}$$

Here

$$Y_j=w_{i+1/2,j}\to Y_j^{K+1},$$

$$X=f_{i+1/2}\to X^{K+1}=\sum_{j=1}^{M}\alpha_j(Y^K)Y_j^{K+1}=\frac{2\pi}{\Theta_{\max}}Rq\left\langle\frac{R}{rJY^k}\right\rangle^{-2}\left\langle\frac{R}{rJ}Y^{K+1}\Big/(Y^K)^2\right\rangle$$

and K is the number of "internal" iterations necessary if $\kappa=0$. By using the above linearization we eliminate X from (6.15) by the equality

$$X=\sum_{j=1}^{M}\lambda_j\Phi_j\Big/\left(1+\sum_{j=1}^{M}\lambda_jQ_j\right), \tag{6.16}$$

where the multipliers $\lambda_j = \lambda_j^k$, as one may easily check, satisfy the system

$$B_{j+1}\lambda_{j+1} - C_j\lambda_j + A_{j-1}\lambda_{j-1} = -\alpha_j, \quad 0 \leqslant j \leqslant M-1,$$
$$\lambda_{j+M} = \lambda_j, \quad A_{j+M} = A_j, \quad B_{j+M} = B_j, \quad j = -1, 0. \tag{6.17}$$

In total, searching for the solution of (6.15) works out into an iterational chain of two cyclic computational sweeps for (6.17) → (6.16) → (6.15). The iterations are terminated when one reaches the accuracy

$$\max_j |(Y_j^{K+1} - Y_j^K)/Y_j^K| \leqslant \epsilon_Y$$

The initial approximation is chosen as $Y_j^0 = w_{i+1/2,j}^n$ or $Y_j^0 = w_{i-1/2,j}^{n+1}$.

6.5. Iteration convergence check

At the present time there is no proof that the iterational process in section 5 converges. However, the numerical experiment shows that it does. We make some remarks on this matter.

An important role in the method proposed is played by the artificial viscosity – the regularizator D. Having D_h in the scheme (6.8), (6.9) provides not only its computational stability but the iterational convergence on the whole as well. Practical computations show that while choosing the constant α_0 in (5.3) it is sufficient for one to restrict oneself by the level

$$\alpha_0 \sim (0.1\text{–}1.0)A,$$

where A is a characteristic scale of coefficients $a_{kl}(x, u)$ in the initial equation.

After the iteration with number n the discussed algorithm defines the grid $\bar{\omega}_h^n$, the functions w_h^n, ρ_h^n and the quantity k^n. Along with w_h^n we introduce the j-independent function

$$\bar{w}^n = k^n v_h.$$

Then on the grid ω_h^n it is natural to consider the function

$$u_h^n \equiv u_i^n = u^0 + \sum_{l=i+1}^{N} \bar{w}_{l-1/2}^n h_{l-1/2}^1$$

to be the solution to the problem (6.1), (6.2). Therefore, as the condition for convergence of the iteration we require the satisfaction of inequalities

$$\| \mathscr{L}_h^n u_h^n + f_h^n \| \leqslant \epsilon_u \| f_h^n \|,$$
$$\| \rho_h^{n+1} - \rho_h^n \| \leqslant \epsilon_\rho. \tag{6.18}$$

Here $\| y_h \| = \| y_h \|_c = \max_{\bar{\omega}_h} | y_h |$.

Together with (6.18) it is interesting to check the accuracies

$$\| w_h^{n+1} - w_h^n \| \leqslant \epsilon_w \| w_h^n \|, \quad \| w_h^n - \overline{w}_h^n \| \leqslant \bar{\epsilon}_w \| w_h^n \|,$$

$$| k^{n+1} - k^n | \leqslant \epsilon_k | k^n |, \quad \| D_h w_h^{n+1} \| \leqslant \epsilon_D \| f_h^n \|.$$

Analyzing the results of computations with $\epsilon_\rho = 10^{-6}$ yields

$$\epsilon_w \sim \bar{\epsilon}_w \sim \epsilon_\rho, \quad \epsilon_k \sim (10^{-2}–10^{-4})\epsilon_\rho,$$

the largest residual

$$\epsilon_u \sim \epsilon_D \sim (10–10^2)\epsilon_\rho$$

proves to be in (6.1).

The number of iterations needed for achieving the accuracy $\epsilon_u \sim (NM)^{-1}$ is nearly in proportion to the grid dimension in one direction, and the total number of arithmetic operations $Q \sim (NM)^{3/2}$.

Lastly, the internal iterations (6.15)–(6.17) corresponding to the case of given distributions $p(\psi)$, $q(\psi)$ converge very rapidly. In particular, on the grid $N = M = 60$ only 2–3 iterations are required for finding the solution with the accuracy $\epsilon_Y \approx 10^{-6}$.

7. Algorithm for the external confinement field computation

7.1. Iterational procedure

We shall give a computational algorithm for solving the problem (2.18) on the determination of the external confinement field that generates an equilibrium configuration with given geometry and physical parameters. By assuming in (2.18) $\sigma_s = \delta_l = 1$ and representing the flux function $\psi(r, z; J)$ in the form of (2.11) we come to the problem of minimizing the functional

$$W(J) = \sum_{s=1}^{m} \left[\psi_{\mathrm{i}}(r_s, z_s; J) + \sum_{l=1}^{K} J_l G(r_s, z_s; R_l, Z_l) - \psi_{\mathrm{p}} \right]^2 + \alpha \left[\sum_{l=1}^{K} J_l^2 + \psi_{\mathrm{p}}^2 \right], \tag{7.1}$$

$$W(J_0) = \min W(J).$$

We recall that the points (r_s, z_s) belong to the given control contour Γ.

Due to the fact that the inherent plasma field $\psi_{\mathrm{i}}(r, z; J)$ implicity depends on the current J the problem (7.1) is nonlinear. We shall apply successive approximations for its solution.

Let the region Ω_{p}^n occupied by plasma be known on the nth step of iterations. Then within Ω_{p}^n one may solve the boundary value problem of equilibrium, posed in section 2, and hence, find the function $\bar{\psi}^n(r, z) = \psi^n(r, z) - \psi_p^n$. By turning further to formulae (2.16), (2.17) we have

$$\psi_i^n(r, z; J^n) = -\int_{\Gamma_p^n} \frac{1}{r'} \frac{\partial \bar{\psi}^n}{\partial n} G(r, z; r', z') \mathrm{d}\tau'$$

$$+\eta^n(r, z)\bar{\psi}^n(r, z),$$

$$\eta^n(r, z) = \begin{cases} 1, & (r, z) \in \Omega_p^n, \\ 0.5, & (r, z) \in \Gamma_p^n, \\ 0, & (r, z) \overline{\in} \Omega_p^n \cup \Gamma_p^n. \end{cases} \tag{7.2}$$

The new approximation J^{n+1} for the current values we obtain by minimizing the functional

$$W(J^{n+1}) = \sum_{s=1}^{m} \left[\psi_i^n(r_s, z_s; J^n) + \sum_{l=1}^{K} J_l^{n+1} G(r_s, z_s; R_l, Z_l) - \psi_p^n \right]^2 + \alpha \left[\sum_{l=1}^{K} (J_l^{n+1})^2 + (\psi_p^n)^2 \right], \tag{7.3}$$

The minimum condition $\partial W / \partial J_l^{n+1} = 0$ results in the linear system of equations

$$\sum_{l=1}^{K} J_l^{n+1} \sum_{s=1}^{m} G(r_s, z_s; R_l, Z_l) G(r_s, z_s; R_k, Z_k) + \alpha J_k^{n+1}$$

$$= -\sum_{s=1}^{m} \left(\psi_i^n(r_s, z_s; J^n) - \psi_p^n \right) G(r_s, z_s; R_k, Z_k), \quad k = 1, 2, \ldots, K. \tag{7.4}$$

The quantity ψ_p^n in (7.3), (7.4) is determined from the conditions (2.5), (2.5′) or from the minimum condition (7.3). Note that when performing the computations one should choose the regularization parameter α that provides the correctness of system (7.4) mainly from the physical considerations. Large currents are unfavourable from the energy point of view, therefore to reduce them one should increase α, which results in removing the plasma boundary Γ_p from the control contour Γ.

After having derived (7.4) we find the next approximation for the region Ω_p^{n+1} from the condition

$$\psi^{n+1/2}(r, z; J^{n+1}) = \psi_i^n(r, z; J^n) + \psi_e^{n+1}(r, z; J^{n+1}) = \psi_p^n, \quad (r, z) \in \Gamma_p^{n+1}, \tag{7.5}$$

that determines the boundary Γ_p^{n+1}.

At the convergence of the above iterational process, formulae (2.11), (2.17) enable us to find the flux function $\psi(r, z; J)$ in the whole unlimited region by solving the internal boundary value problem of equilibrium.

Note that in the direct case the iterational procedure (7.2), (7.5) with $J^n \equiv J$ coincides with the successive approximation technique usually used for calculation of the axisymmetric equilibrium in external fields with a limiter [5].

7.2. Numerical technique

The main purpose of the iteration process under discussion is the solution of the internal equilibrium problem and the computation of the contour integral (7.2) at some points, which is necessary for obtaining new approximations of currents and plasma boundary from (7.3)–(7.5).

The first purpose may be achieved by using the algorithm given in section 6. The approximation of the contour integral (7.2) will be discussed later one. As for the numerical implementation of (7.5) we shall act in the following way. At a small distance from the boundary Γ_{p}^n (by stretching the rays $\Theta(r, z) = \mathrm{const}$) we choose two contours γ_1^n and γ_2^n respectively, inside and outside Ω_{p}^n. At the points obtained we determine the function $\psi^{n+1/2}(r, z; J^{n+1})$ by linear interpolation (or by extrapolation with restriction) and obtain the contour $\Gamma_{\mathrm{p}}^{n+1}$ where the condition (7.5) is approximately satisfied.

The computation results have shown that the iterational process described converges rather rapidly. On the grid $N = 20$, $M = 60$ inside the plasma Ω_{p}^n and $m = 20$ from 10 to 20 iterations are required to determine the currents and the boundary Γ_{p}^n with a relative accuracy 10^{-5}–10^{-6}.

The Green's function (2.12) has a logarithmic singularity for

$$P = (r, z) \to (r', z') = P',$$

$$G(P, P') = \frac{\sqrt{rr'}}{2\pi}\left[\bar{G}(P, P') - \ln \rho(P, P')\right], \tag{7.6}$$

$$\rho(P, P') = \sqrt{(r - r')^2 + (z - z')^2}.$$

Let the boundary Γ_{p} is approximated by a broken line $\Gamma_{\mathrm{p},h}$ with apexes $P_j = (r_j, z_j)$, $j = 1, 2, \ldots, M$. Then the contour integral in (2.17) is calculated with exact allowance for the singularity (7.6) for each segment $\Gamma_{j-1/2}$ of the broken line $\Gamma_{\mathrm{p},h}$, namely:

$$\int_{\Gamma_{\mathrm{p}}} \frac{1}{r'} \frac{\partial \psi}{\partial n} G(r, z; r', z')\mathrm{d}\tau' \simeq \sum_{j=1}^{M} \sigma(P, P_{j-1/2})\left[\bar{G}(P, P_{j-1/2}) - F(P, P_{j-1/2})\right]. \tag{7.7}$$

Here

$$\sigma(P, P') = \frac{\sqrt{rr'}}{2\pi} \frac{1}{r'} \frac{\partial \psi}{\partial n}(r', z')\delta\tau_{j-1/2}$$

$$\delta\tau_{j-1/2} = \rho(P_j, P_{j-1}), \quad P_{j-1/2} = \left(\frac{r_j + r_{j-1}}{2}, \frac{z_j + z_{j-1}}{2}\right), \tag{7.8}$$

$$F(P, P_{j-1/2})\delta\tau_{j-1/2} = \int_{\Gamma_{j-1/2}} \ln \rho(P, P')\mathrm{d}\tau'.$$

Note that if the point P lies within a ϵ-vicinity of the point $P_{j-1/2}$ the function $\bar{G}$ cannot be calculated by formula (7.6). In this case

$$\rho(P,\ P_{j-1/2}) \leqslant \epsilon\delta\tau_{j-1/2}$$

Let

$$\bar{G}(P,\ P_{j-1/2}) = 0.5(\bar{G}(P,\ P_j) + \bar{G}(P,\ P_{j-1})).$$

The integral (7.8) is evaluated in terms of the elementary functions

$$F(P,\ P_{j-1/2})\delta\tau_{j-1/2} = \left[d_{j-1/2} \operatorname{arctg}\left(\frac{\tau - \tau_{j-1/2}}{d_{j-1/2}}\right) + (\tau - \tau_{j-1/2}) \times \left[\tfrac{1}{2}\ln\left(d_{j-1/2}^2 + (\tau - \tau_{j-1/2})^2\right) - 1\right]\right]_{\tau=0}^{\tau=\delta\tau_{j-1/2}}, \tag{7.9}$$

$$d_{j-1/2} = |(r - r_{j-1})(z_j - z_{j-1}) - (z - z_{j-1})(r_j - r_{j-1})|/\delta\tau_{j-1/2},$$

$$\tau_{j-1/2} = ((r - r_{j-1})(r_j - r_{j-1}) + (z - z_{j-1})(z_j - z_{j-1}))/\delta\tau_{j-1/2}.$$

For calculations with the aid of (7.7)–(7.9) the normal derivative $\partial\psi/\partial n$ in the solution of the elliptic problem must be approximated for Γ_{p}. The approximation is obtained by assuming $\psi(r,\ z)$ to be a piecewise bilinear function at the quadrange $\Omega_{N-1/2,j-1/2}$ adjacent to the boundary $\Gamma_{\mathrm{p},h}$ and having the apexes where it takes the values ψ_h determined by the solution of difference problem (6.1), (6.2). More exact approximations of $\partial\psi/\partial n$ may be constructed by the method proposed in ref. [33].

Formula (7.7), which provides higher accuracy, is used also for determining $\psi(r,\ z)$ in a calculated rectangle that contains the plasma (in order to complete construction of magnetic surfaces); outside the rectangle it is more natural to apply a usual quadric formula.

8. On tests in MHD equilibrium problems

We shall give some solutions to the equilibrium eq. (2.2), which are convenient for using as tests to check the accuracy of computational algorithms.

The function

$$\psi(r,\ z) = c_0[(\Delta^2 - r^2)(r^2 - \delta^2) - 4\alpha^2(r^2 - \sigma^2)z^2] \tag{8.1}$$

satisfies [34] the equilibrium equation for

$$\frac{\mathrm{d}p}{\mathrm{d}\psi} = 8(1 + \alpha^2)c_0,$$
$$\frac{\mathrm{d}f^2}{\mathrm{d}\psi} = -16\alpha^2\sigma^2 c_0, \quad c_0 = 4\psi_{\max}/(\Delta^2 - \delta^2)^2. \tag{8.2}$$

This solution corresponds to the safety factor

$$q(\psi) = q_m[1 + \lambda(1 - \psi/\psi_{max})]^{1/2} Q(\psi), \quad q_m = q(\psi_{max}),$$
$$\lambda = 0.5 q_m^{-2} \sigma^2 (\Delta^2 - \delta^2)^2 (\Delta^2 + \delta^2)^{-2} (\Delta^2 + \delta^2 - 2\sigma^2)^{-1},$$
$$Q(\psi) = \frac{1}{\pi} \int_0^\pi (1 + \omega(\psi) \cos\Theta)^{-1} (1 + \omega_0 \omega(\psi) \cos\Theta)^{-1/2} d\Theta, \tag{8.3}$$
$$\omega(\psi) = (\Delta^2 - \delta^2)/(\Delta^2 + \delta^2)(1 - \psi/\psi_{max})^{1/2}, \quad \omega_0 = (\Delta^2 + \delta^2)(\Delta^2 + \delta^2 - 2\sigma^2)^{-1}.$$

The functions $p(\psi)$, $f(\psi)$ are given by

$$p(\psi) = 2c_0^2(1 + \alpha^2)(\Delta^2 - \delta^2)^2 \psi/\psi_{max},$$
$$f(\psi) = f_m[1 + \lambda(1 - \psi/\psi_{max})]^{1/2}, \tag{8.4}$$
$$f_m = f(\psi_{max}) = 2\sqrt{2}\, c_0 \alpha q_m (\Delta^2 + \delta^2)(\Delta^2 + \delta^2 - 2\sigma^2)^{1/2}.$$

The plasma boundary Γ_p coincides with the casing Γ_0 and is obtained from the condition $\psi(r, z) = 0$. The parameters $\alpha > 0$, $0 \leqslant \sigma \leqslant \delta < \Delta$ define a geometry of the plasma cross-section so that

$$\Delta = r_{max} = \max_{(r,z)\in\Gamma_0} r, \quad \delta = r_{min} = \min_{(r,z)\in\Gamma_0} r.$$

The main geometric characteristics of the plasma are: the major torus radius $R = 0.5(\Delta + \delta)$, the aspect ratio

$$A = \frac{2R}{r_{max} - r_{min}} = \frac{\Delta + \delta}{\Delta - \delta},$$

the plasma elongation

$$E = \frac{z_{max} - z_{min}}{r_{max} - r_{min}} = \frac{1}{\alpha}\left[\Delta^2 + \delta^2 - 2\sigma^2 - 2\left((\Delta^2 - \sigma^2)(\delta^2 - \sigma^2)\right)^{1/2}\right]^{1/2} (\Delta - \delta)^{-1},$$

the triangularity of the plasma boundary ($r^* = r(z_{max})$) (8.5)

$$G = \frac{R - r^*}{R - r_{min}} = \left[\Delta + \delta - 2\left[\sigma^2 + \left((\Delta^2 - \sigma^2)(\delta^2 - \sigma^2)\right)^{1/2}\right]^{1/2}\right](\Delta - \delta)^{-1},$$

the magnetic axis coordinates

$$(r^m, z^m) = \left([0.5(\Delta^2 + \delta^2)]^{1/2}, 0\right),$$

the relative axis shift

$$\xi = \frac{r^{\mathrm{m}} - R}{r_{\max} - R} = \left(\left[2\left(1 + \frac{\delta^2}{\Delta^2}\right) \right]^{1/2} - \left(1 + \frac{\delta}{\Delta}\right) \right) \left(1 - \frac{\delta}{\Delta}\right)^{-1} \leqslant \sqrt{2} - 1,$$

the stretching of magnetic surfaces near the axis

$$\epsilon = \frac{1}{\alpha} (\Delta^2 + \delta^2)^{1/2} (\Delta^2 + \delta^2 - 2\sigma^2)^{-1/2} \leqslant E.$$

Lines of the level $\psi(r, z) = \text{const}$ from (8.1) may be presented in the parametric form

$$\begin{aligned} &r(a, \Theta) = r^{\mathrm{m}} [1 + \omega(a) \cos \Theta]^{1/2}, \\ &z(a, \Theta) = 0.5 \alpha^{-1} \left[r^2(a, \Theta) - \sigma^2 \right]^{-1/2} (r_m)^2 \omega(a) \sin \Theta, \\ &\omega(a) = (\Delta^2 - \delta^2)(\Delta^2 + \delta^2)^{-1} a^{1/2}, \\ &a = 1 - \psi/\psi_{\max} \in [0, 1], \quad \Theta \in [0, 2\pi]. \end{aligned} \tag{8.6}$$

On the magnetic axis $(r^{\mathrm{m}}, z^{\mathrm{m}})$ we have the quantities

$$\beta_{\mathrm{p}} = r\frac{\mathrm{d}p}{\mathrm{d}\psi} \bigg/ \left(r\frac{\mathrm{d}p}{\mathrm{d}\psi} + \frac{1}{2r}\frac{\mathrm{d}f^2}{\mathrm{d}\psi} \right) = 1 + \frac{2\sigma^2}{(\alpha^{-2} + 1)(\Delta^2 + \delta^2 - 2\sigma^2)}, \tag{8.7}$$

$$\beta = \frac{r^2 p}{f^2} = \tfrac{1}{8}(1 + \alpha^{-2}) q_{\mathrm{m}}^{-2} \left(\frac{\Delta^2 - \delta^2}{\Delta^2 + \delta^2} \right)^2 \frac{\Delta^2 + \delta^2}{\Delta^2 + \delta^2 - 2\sigma^2}.$$

9. Computational examples

As computation practice shows the algorithm discussed here may efficiently be applied to solving a wide range of problems which are dealt with the evolution and stability of equilibrium plasma configurations with axial symmetry. In this section capabilities of the technique are demonstrated on a series of specific examples. In subsection 9.1 comparisons are given between the exact solution (8.1)–(8.7) to the equilibrium equation and the numerical experiment carried out in quasipolar flux coordinates. The solution of the evolution problem for an ideally conducting plasma under an increasing pressure is considered in section 9.2. This example is rather exotic due to the smallness of aspect ratio ($A = 1.5$) and large values of q ($2 \leqslant q \leqslant 14$) but it is interesting in that it shows strong deformation of the magnetic surface system. The next two subsections are concerned with demonstration of orthogonal inverse variables being applied to two evolution problems. The case in 9.3 on the so called plasma "exhaust" arises in connection with the basic problem concerning tokamak-reactors – the cleanup of working chambers from

Table 1

Case	1	2	3	4
R	5.2	5.2	5.2	1.95
A	4	4	4	1.5
E	1.5	2.0	1.5	1.5
G	0.3	0.5	0.9	0.5
exact r^{m}	5.360	5.360	5.360	2.344
exact ψ_{max}	1.0	1.0	1.0	1.0
calculated r^{m}	5.360	5.360	5.359	2.343
calculated ψ_{max}	0.9977	0.9977	0.9967	0.9967

burning products and accessory contaminants. Computations [20] confirm the idea proposed in ref. [35] that the exhaust occurs due to the ballooning effect. In 9.4 we determine possible equilibrium plasma states in compact tori limited in stability with respect to an interchange mode – a most dangerous one in such traps [36]. This problem is a particular case of the adiabatic equilibrium problem with $\eta(\psi) = \text{const}$, which was studied in detail in ref. [24]. The results of computations (in quasipolar coordinates) carried out for the external confining field in INTOR configurations and earlier published in ref. [21,27] are given in 9.5.

9.1. Comparison with tests

Everywhere below we take the function

$$a(\psi) = 1 - \psi/\psi_{max} \in [0, 1]$$

as the basic flux variable, and the constant c_0 in (8.2) is assumed to be

$$c_0 = 4(\Delta^2 - \delta^2)^{-2},$$

which corresponds to $\psi_{max} = 1$. The equilibrium problem is solved for given $p(\psi)$ and $f(\psi)$ by using the expressions (8.2) for different values of parameters (8.5) describing the plasma geometry (aspect ratio, elongation, triangularity). Table 1 and fig. 2 give the results of comparison between the numerical solution obtained on the grid

$$\omega_h' = \Big\{ a_0 = 0,\ a_i = \sqrt{(i - 0.5)/(N - 0.5)}\,, 1 \leqslant i \leqslant N;$$

$$\Theta_j = 2\pi j/M, 0 \leqslant j \leqslant M \Big\}, \quad N = 19,\ M = 64,$$

and the exact solution constructed by parametric formulae (8.6). We give here every second magnetic surface beginning from the first one. As is seen from fig. 2 the exact and numerical isolines $\psi(r, z) = \text{const}$ practically coincide, a relative accuracy of their determination being about 10^{-4}–10^{-5}. In the case of large triangularity (table 1-3) some difference is attributed mainly to a choice of points $\Theta(r, z) = \text{const}$ in parametrization (8.6). Fig. 3 shows the comparison results for given functions (8.3), (8.4), $p(\psi)$ and $q(\psi)$, for $N = 19$, $M = 20$, and $N = 59$, $M = 60$, $a_0 = 0$, $a_i = (i - 0.5)/(N - 0.5)$, $1 \leqslant i \leqslant N$. Here $\psi_{max} = 1$, $q_m = 2.0$, $(2 \leqslant q(a) \lesssim 2.9)$, the geometric parameters are listed in table 1-1.

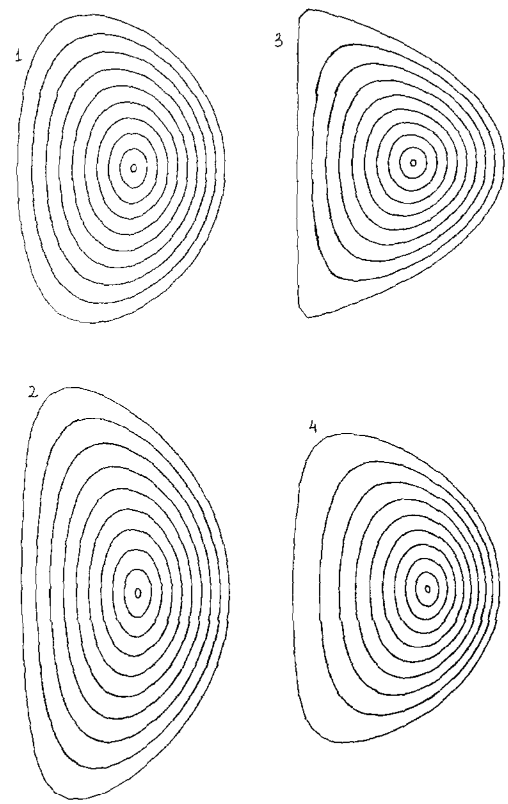

Fig. 2. Comparison between exact and numerical solutions at given $p(\psi)$ and $q(\psi)$ in quasipolar flux coordinates (table 1).

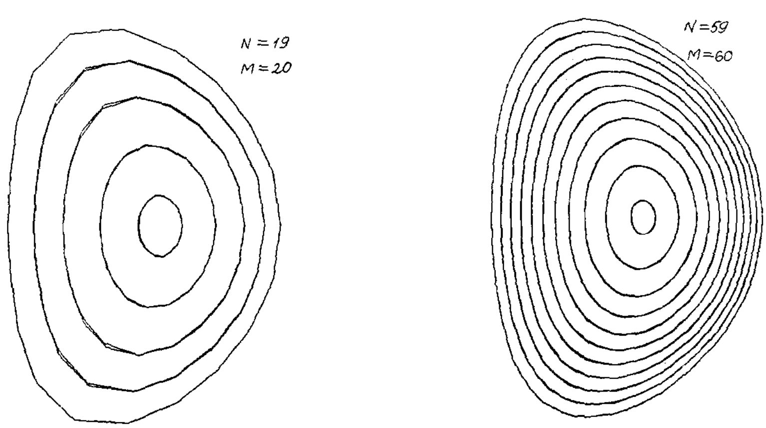

Fig. 3. Comparison between exact and numerical solutions at given $p(\psi)$ and $q(\psi)$ on different grids (quasipolar coordinates).

By analyzing the results we may conclude that the algorithm for quasipolar inverse coordinates yields high accuracy in determining the magnetic surfaces and other characteristics of equilibrium plasma configurations, namely $q(\psi)$, $f(\psi)$, $\psi_{\max}$ and the magnetic axis shift. The typical time for computing the equilibrium on the grid with $N = M = 60$ is about 3 min on the BESM-6 computer (one iteration in the process of sections 5.6 takes 3 s).

As for the technique of the orthogonal ($\gamma = 1$) flux coordinates, we note that when the surfaces $\psi(r, z) = \text{const}$ are elongated $\epsilon \neq 1$ near the axis, it proves to be inferior to the quasiorthogonal coordinate technique. It may be explained by "sticking" of isolines $\Theta(r, z) = \text{const}$ near the axis, which results in strong irregularity of the Eulerian grid $\bar{\omega}_h = \{r_{ij}, z_{ij}\}$. One may eliminate this "sticking" and achieve higher accuracy be properly choosing the quasiorthogonal coordinates with $\gamma \sim \epsilon^{-1}$. Nevertheless, the solution of the problem obtained by using the orthogonal variables yields rather feasible geometric and physical parameters of the equilibrium plasma. Moreover, the technique has a wide range of applications since it does not require the starwise condition for magnetic surfaces which is necessary in the case of direct (without a preliminary replacement of variables) introduction of quasipolar coordinates.

9.2. Evolution of an ideally conducting plasma

This problem (see section 2.3) was solved by the quasipolar coordinate algorithm for the fix shaped plasma (table 1-4). The flux coordinates and the computational grid were chosen like in 9.1. The safety factor was given by (8.3)

$$q_m = 2.0, \quad \psi_{\max} = 1, \quad (2.0 \leqslant q \lesssim 14.0)$$

and the pressure

$$p(\psi) = 8p_0(1 + \alpha^2)c_0\psi.$$

The magnetic surfaces obtained by computations for different pressures (parameter p_0) are shown in fig. 4. The respective plasma characteristics are listed in table 2 including the values that define the tokamak energy performance:

$$\beta = \left(\frac{\int_{\Omega_p} p^2 \mathrm{d}S}{S_p} \right)^{1/2} \frac{S_p}{\int_{\Omega_p} B_\varphi^2 \mathrm{d}S} \approx \beta^* = \frac{2\int_{\Omega_p} pr \mathrm{d}S}{\int_{\Omega_p} B_\varphi^2 r \mathrm{d}S},$$

$$\beta_p = \int_{\Omega_p} \frac{\mathrm{d}p}{\mathrm{d}\psi} r \mathrm{d}S / J_p \approx \beta_J = 8\pi \int_{\Omega_p} p \mathrm{d}S / J_p^2,$$

$$\beta_I = \frac{1}{4\pi} \frac{\mathscr{L}_p^2}{S_p} \beta_J \approx \beta_J, \quad S_p = \int_{\Omega_p} \mathrm{d}S, \; \mathscr{L}_p = \int_{\partial\Omega_p} \mathrm{d}l,$$

$$J_p = \int_{\Omega_p} \left(r \frac{\mathrm{d}p}{\mathrm{d}\psi} + \frac{1}{2r} \frac{\mathrm{d}f^2}{\mathrm{d}\psi} \right) \mathrm{d}S, \quad B_\varphi = \frac{f}{r}.$$

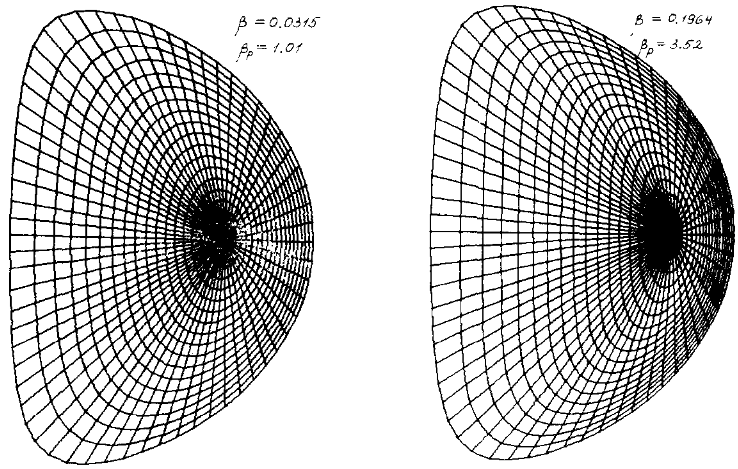

Fig. 4. Equilibrium for flux conserving evolution. Plasma touches the casing (table 2).

From the computations it is seen that when the pressure increases due to the ballooning effect the magnetic surfaces are more elongated, the axis shifts outside, and the plasma diamagnetism (the difference $f(a_{max}) - f(0)$) grows. High values of $\beta \approx 20\%$, $\beta_p \approx 3.5$ are reached even for $q \approx 2$–14.

9.3. "Exhaust" problem solution

Let the plasma fill the working chamber of an ideally conducting two-chamber casing shown in fig. 5. In order to evaluate the possibility of exhaust due to the ballooning effect one should determine the parameters for which the plasma ceases being maintained in the working chamber and flows through a rather narrow gap into an auxiliary chamber. This problem is solved in the evolution model (2.3). In accordance with the conditions (2.4)–(2.7) we give the values'

$$\psi_{max} = 1, \quad \psi_p = \tfrac{2}{3}, \quad \psi_0 = 0$$

Table 2

p_0	1	2	4	6
β (%)	3.15	6.41	13.01	19.64
β^* (%)	4.01	8.42	17.75	27.50
β_p	1.01	1.72	2.75	3.52
β_J	0.81	1.33	1.76	1.98
J_p	12.97	15.19	18.99	22.25
$f(0)$	5.530	5.317	4.862	4.377
$f(a_{max})$	5.533	5.587	5.698	5.815
r^m	2.343	2.491	2.644	2.722

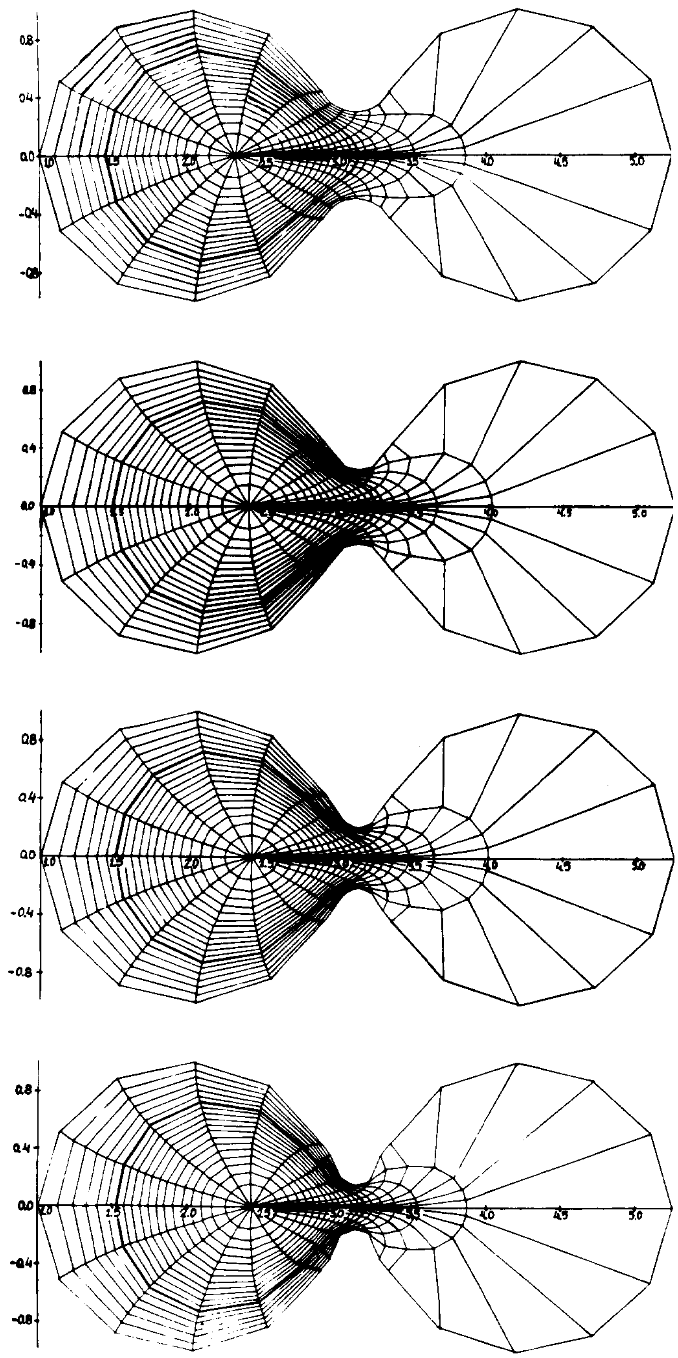

Fig. 5. "Exhaust": Series of limited equilibria ($p_m \approx p_m^c$) at different width of gap d (table 3).

Table 3

d	0.30	0.25	0.20	0.15
β (%)	2.0	3.3	4.1	5.1
p_m	0.34	0.400	0.67	0.90

and the distributions

$$p(a) = p_m(1 - a/a_p),$$

$$q(a) = q_m + (q_p - q_m)a/a_p,$$

$$a(\psi) = 1 - \psi/\psi_{max}, \quad q_m = 1, \quad q_p = 1.67.$$

The solution of this problem in direct Eulerian coordinates (r, z) meets some difficulties. The latter are due to a complex geometry of the region, a free boundary, and a nonlinear integro-differential structure of operator (2.9). Besides, at small pressures the problems admits two solutions, i.e. the plasma may occupy either the left or right chamber. The inverse variable technique based on introduction of orthogonal coordinates allows one to overcome the above difficulties.

The purpose of computations consisted in finding the critical value p_m^c and the respective value β^c. When these values are exceeded there is only one solution – the plasma equilibrium in the auxiliary chamber. Table 3 and fig. 5 give a number of the system states close to critical ones (and with acceptable β) in dependence on the relative width of the gap d. It can be seen that for small d an outer contour of the plasma has a typical peak shape directed into the gap between the chambers.

9.4. Adiabatic equilibrium in compact tori

As discussed above, searching for equilibrium plasma states in compact tori with limited stability with respect to the interchange mode is equivalent to the solution of the adiabatic evolution problem in 2.4. We give its computations with parameters

$$\psi_{max} = 1, \quad \psi_p = \psi_0 = 0, \quad df^2/d\psi = 0,$$

$$\eta(\psi) = c, \quad \gamma = \tfrac{5}{3}.$$

Ellipsoidal configurations are given in fig. 6 for the fixed plasma volume $V_p = 4\pi/3$ and the quantity

$$\int_{V_p} p^2 d\tau = c^2 \int_{V_p} \left(- \frac{d\psi}{dV} \right)^{\gamma} d\tau = \text{const}$$

which characterizes the intensity of thermonuclear reactions, $c = 8.8$ for a sphere. It is shown that

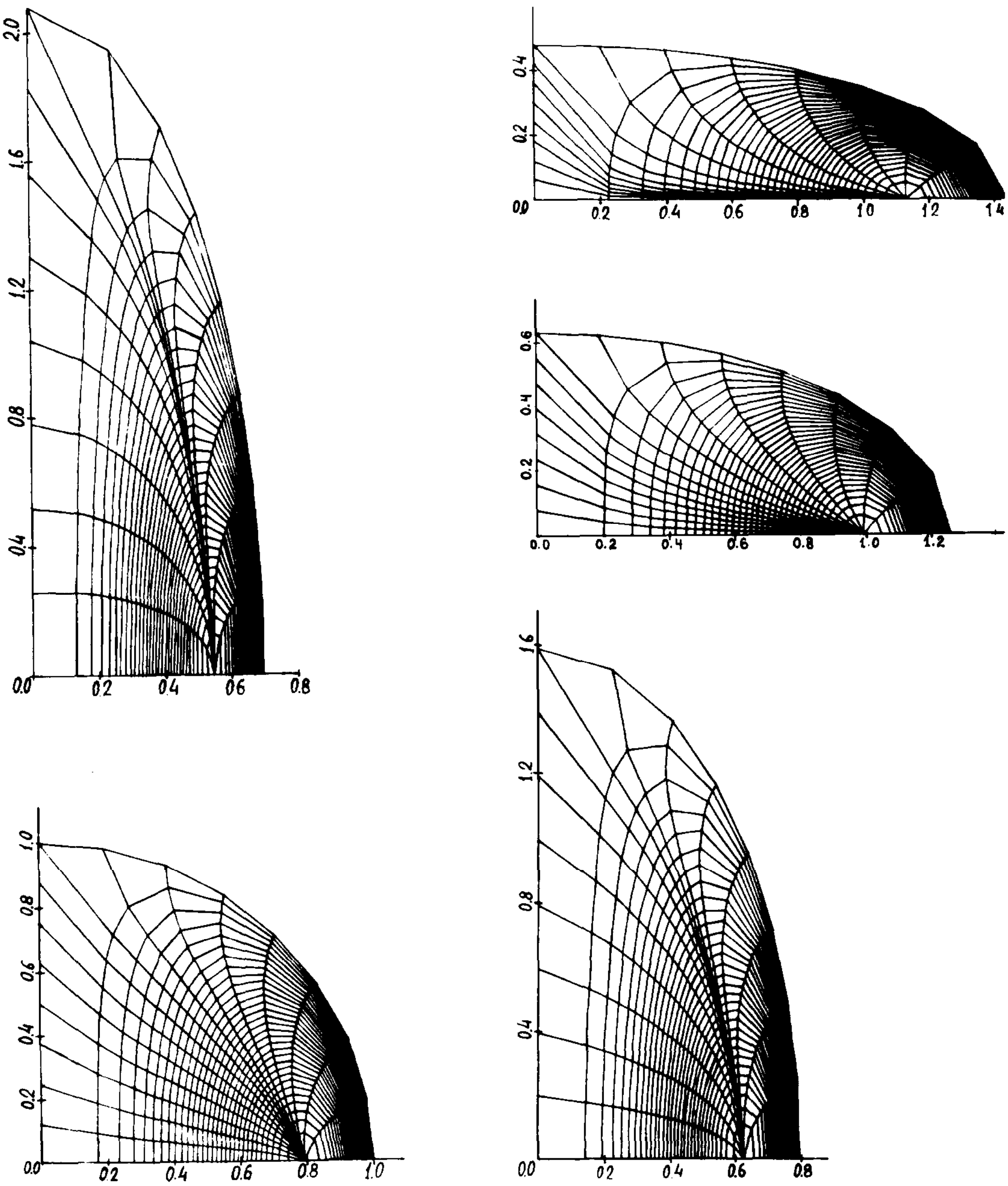

Fig. 6. Equilibrian compact tori with limited stability against interchange modes. Plasma occupies ellipsoids with equal volume at $\int_{V_p} p^2 \mathrm{d}V = \mathrm{const.}$

in such equilibrium configurations stability limited against interchange disturbances pressure may be strongly peaked $p_{max}/p_{min} \approx 100$. The magnetic axis, however, is removed from the symmetry axis $r = 0$ so that the plama is practically a torus with large aspect ratio $A \approx 4$.

9.5. *Calculation of external fields in INTOR like configurations*

Consider an example of the algorithm discussed in section 7 in application to computation of an external confining field in the INTOR like plasma configuration. The control contour Γ to

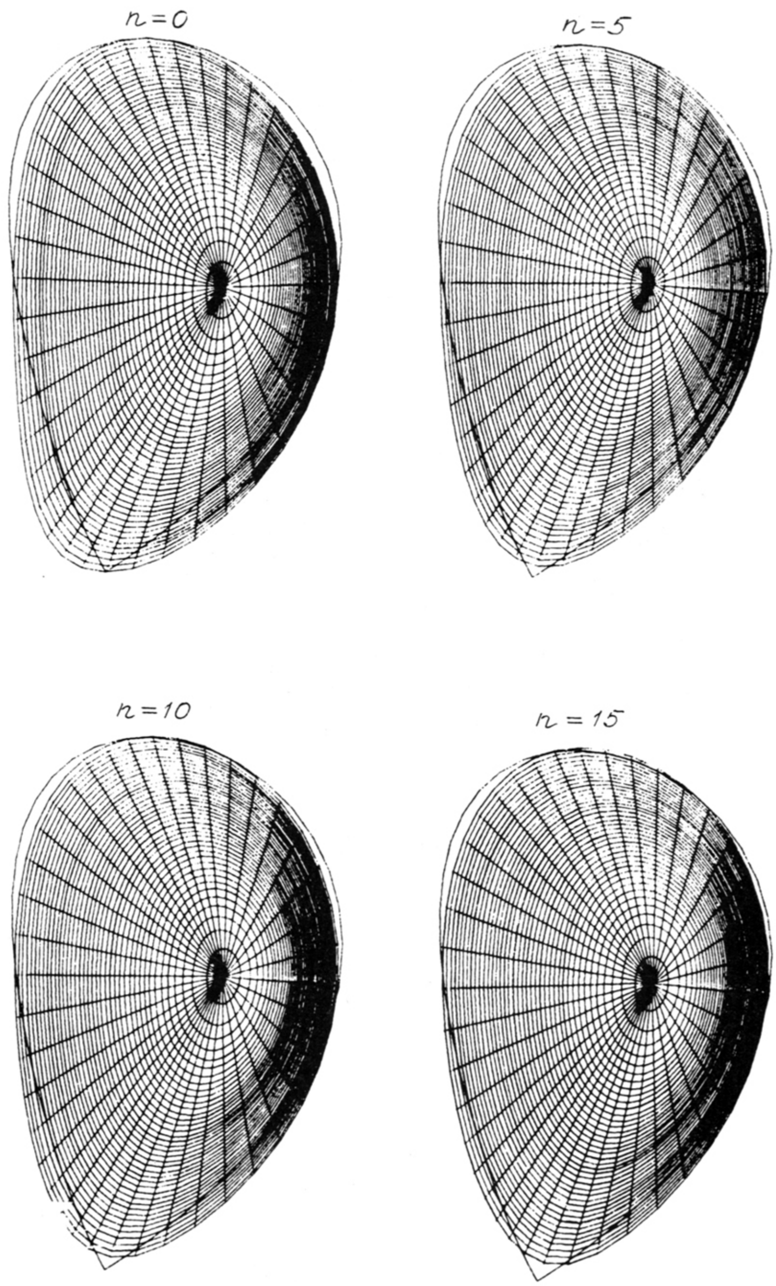

Fig. 7. Iterations of plasma boundaries in calculations of external field in INTOR like configurations (Γ is a control contour).

Table 4

β	β_I	β_J	J_p (MA)		Coil currents (MA)									$\Sigma_{l=1}^{9} \lvert J_l \rvert$
				Number of coils:	1	2	3	4	5	6	7	8	9	
				R (m):	10.3	2.6	1.4	1.4	1.4	4.0	6.0	10.3	10.8	
				Z (m):	5.5	6.4	3.6	0.2	−3.9	−7.1	−7.1	−5.6	−5.0	
0.3%	0.24	0.2	5.7		−1.8	6.0	5.4	−21.4	7.3	13.2	13.6	−6.4	−9.5	84.6
1.8%	1.1	0.9	6.9		−2.6	8.4	4.7	−19.2	6.0	13.5	15.5	−6.4	−11.0	87.3
5.6%	2.6	2.2	6.4		−4.4	10.2	4.1	−18.5	5.9	15.2	18.3	−7.8	−14.2	98.6

which the plasma boundary Γ_p must be drawn was chosen in the form

$$r(\varphi) = R + \frac{R}{A}\left[\left(\frac{1+\sin\varphi}{D+\sin\varphi}\right)^{\gamma}\cos\varphi - 0.5(G_1 + G_2 + (G_1 - G_2)\sin\varphi)\sin^2\varphi\right],$$

$$z(\varphi) = Z + (R/A)0.5[E_1 + E_2 + (E_1 - E_2)\sin\varphi]\sin\varphi,$$

$$\varphi \in [0, 2\pi],$$

with the typical INTOR parameters $R = 5.3$, $Z = 0.6$ M, $A = 5.3/1.2$, $E_1 = 1.5$, $E_2 = 1.7$, $G_1 = 0.2$, $G_2 = 0.4$. At $\varphi = \frac{3}{2}\pi$ the contour Γ has a corner imitating a separatrix, it is controlled by the quantities $D = 1.3$ and $\gamma = 0.5$. The pressure and safety factor distributions over the magnetic surfaces were given as

$$p(a) = p_m(1-a)^2, \quad q(a) = q_m + (q_p - q_m)a^2,$$

$$a = 1 - (\psi - \psi_p)(\psi_{max} - \psi_p)^{-1},$$

$$q_m = 1.0, \quad q_p = 2.1, \quad \psi_{max} - \psi_p = 1.0.$$

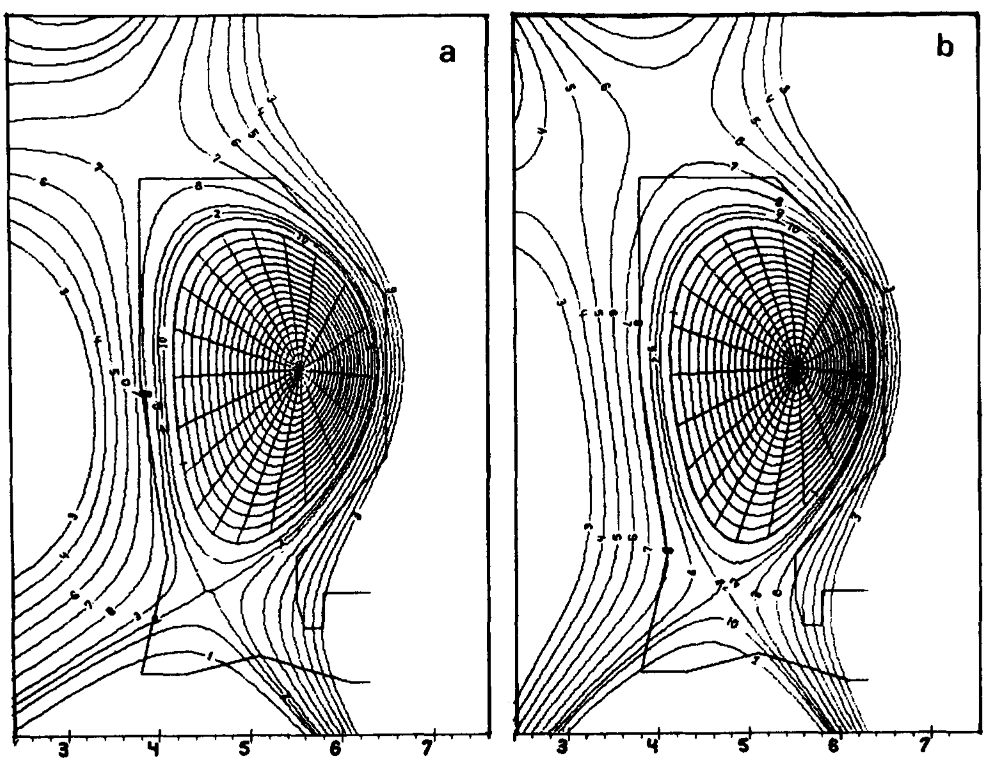

Fig. 8. INTOR like plasma configurations; (a) table 4, (b) table 5.

Table 5

β	β_I	β_J	J_p (MA)		Coil currents (MA)										$\Sigma_{l=1}^{10} \mid J_l \mid$
				Number of coils:	1	2	3	4	5	6	7	8	9	10	
				R (m):	11.15	4.7	3.8	2.5	1.35	1.35	3.0	4.7	7.2	12.1	
				Z (m):	5.1	7.1	7.0	6.6	2.55	−1.95	−6.9	−7.1	−7.0	−5.4	
5.6%	2.6	2.2	6.6		−2.1	−0.5	6.8	10.5	11.6	−1.2	22.0	27.1	4.9	−25.4	111

The dimension quantities were recalculated from the condition that the longitudinal field $B_\varphi = 5.5$ T at the chamber center (R, Z).

In computations the value of ψ_p was obtained by minimizing the functional (7.1), and the regularization parameter $\alpha = 0.001$ was chosen so that the configuration must satisfy certain engineering requirements (a proper geometry of the divertor channel, reasonable currents, etc.).

Fig. 7 illustrates the convergence of iterational process in section 7 ($n = 0, 5, 10, 15$), $m = 20$, $N = 39$, $M = 40$. The equilibrium configuration with 9 confining coils was studied in detail (table 4). It is obtained by a crude optimization in the number and position of external currents and shown in fig. 8a for the maximal $\beta \approx 5.6\%$. The values β, β_I are obtained by integration over the cross section of the region Ω_s bounded by the separatrix. The poloidal field and currents for the INTOR type arrangement of coils are shown in fig. 8b and table 5. Here the internal separatrix branch is declined downward at a larger distance which allows more convenient arrangement of divertor plates.

Analysis of computations shows that the formation of the INTOR like plasma configuration requires a small (≈ 10) number of confining coils while a total current in coils must be $\sum_{l=1}^{K} |J_l| \approx 100$ MA at the plasma current $J_p \approx 6.4$ MA.

10. Conclusions

Aside from the above examples the algorithm described has been applied successfully to solving the following MHD-equilibrium problems.

1. Optimization of the helical system parameters on the basis of an exact 2-D equilibrium equation [23,26] and a stellarator approximation [22].
2. Investigation of the stability against the ideal MHD mode by means of 1-D criteria [25] and the solution of a complete spectral problem.
3. Searching for equilibrium configurations with given $q(\psi)$ and limiting stability against all ideal MHD modes [37].
4. Study of transport processes in a 1.5 D-scale model [27].

In all the cases the inverse variable algorithms proved to be a convenient and efficient means of computations.

At the present time the quasipolar flux coordinate case is being generalized for the solution of 3-D (nonsymmetric) MHD equilibrium problem in the exact scalar statement [38,39] similar to the Grad–Shafranov equation.

Acknowledgements

We are grateful for the encouragement and continuing support of A.A. Samarskii and V.D. Shafranov. We are indebted also P.N. Vabishchevich, S.A. Galkin, L.P. Lobanova, A.A. Martynov, S.Yu. Medvedev and Yu.Yu. Poshekhonov for cooperation in development of inverse variables technique and its application to different MHD problems.

References

[1] V.D. Shafranov, JETP 33 (1957) 710.
[2] L.E. Zakharov and V.D. Shafranov, in: Plasma Theory Questions, vol. 11, eds. M.A. Leontovich and B.B. Kadomtsev (Energoizdat, Moscow, 1982) p. 118.
[3] R.A. Dory and Y.K. Peng, Nucl. Fusion 17 (1977) 21.
[4] H. Grad, P.N. Hu and D.C. Stevens, Proc. U.S. Nat. Acad. Sciences, 72 (1975) 3789.
[5] K. Lacner, Comput. Phys. Commun. 12 (1976) 33.
[6] J.L. Johnson, H.E. Dalhed, J.M. Green et al., J. Comput. Phys. 32 (1979) 212.
[7] S. Hamada, Nucl. Fusion 1 (1962) 23.
[8] J.M. Green and J.L. Johnson, Phys. Fluids 5 (1962) 510.
[9] G. Bateman, Nucl. Fusion 13 (1973) 227.
[10] V.D. Khait, Fizika Plazmy 6 (1980) 871.
[11] L.L. Lao, S.P. Hirchman and R.M. Wieland, Phys. Fluids 24 (1981) 1431.
[12] J. Delucia, S.C. Jardin and A.M.M. Todd, J. Comput. Phys. 37 (1980) 183.
[13] D.E. Shumaker, J. Comput. Phys. 53 (1984) 456.
[14] P.N. Vabishchevich, L.M. Degtyarev and A.P. Favorsky, Fizika Plazmy 4 (1978) 995.
[15] P.N. Vabishchevich and L.M. Degtyarev, Dokl. AN SSSR 247 (1979) 1342.
[16] P.N. Vabishchevich, L.M. Degtyarev and Yu.Yu. Poshekhonov, J. Comput. Math. Math. Phys. 20 (1980) 492.
[17] P.N. Vabishchevich, L.M. Degtyarev and V.V. Drozdov, Voprosy Atomnoi nauki i tekhniki, Series: Metodiki i programmy chislennogo resheniya zadach mat. fiziki 2 (1982) 22.
[18] P.N. Vabishchevich, L.M. Degtyarev and V.V. Drozdov, Dokl. AN SSSR 262 (1982) 1040.
[19] L.M. Degtyarev and V.V. Drozdov, Differentsial'nye uravneniya, 20 (1984) 1194.
[20] L.E. Zakharov, M.I. Mikhajlov, V.I. Pistunovich et al., Proc. 8th Intern. Conf. on Plasma Phys. and C.N.F.R., IAEA, Vienna 1 (1981) 313.
[21] S.G. Bespoludennov, L.E. Zakharov, V.I. Pistunovich et al., Proc. 10th Europ. Conf. on Plasma Phys. and C.N.F.R. 1 (1981) G-4.
[22] L.E. Zakharov, M.I. Mikhajlov, V.D. Pustovitov et al., Proc. 9th Intern. Conf. on Plasma Phys. and C.N.F.R. (1982).
[23] L.M. Degtyarev, Yu.Yu. Poshekhonov and V.D. Shafranov, Proc. 11th Europ. Conf. on Plasma Phys. and C.N.F.R. (1981).
[24] P.N. Vabishchevich, L.M. Degtyarev, V.V. Drozdov et al., Fizika Plazmy 7 (1982) 973.
[25] P.N. Vabishchevich, L.M. Degtyarev and S.Yu. Medvedev, Fizika Plasmy 9 (1983) 163.
[26] P.N. Vabishchevich, L.M. Degtyarev, Yu.Yu. Poshekhonov et al., Fizika Plasmy 9 (1983) 481.
[27] S.G. Bespoludennov, P.N. Vabishchevich, L.M. Degtyarev et al., Preprint of the Keldysh Inst. of Appl. Math., AN SSSR N 30 (1983).
[28] F. Bauer, O. Betancourt and P. Garabedian, A Computational method in Plasma Physics, Springer Series in Computational Physics (Springer Verlag, Berlin, 1978).
[29] Jh.L. Lions, Optimal Control of Systems Described by partial Differential Equations (Mir, Moscow, 1972).
[30] A.N. Tikhonov and V.Ya. Arsenin, Methods of Solution of Incorrect Problems (Nauka, Moscow, 1974).
[31] S.G. Mikhlin, Variational Methods of Mathematical Physics (Nauka, Moscow, 1969).
[32] A.A. Samarsky and V.B. Andreev, Difference Methods for Elliptic Equations (Nauka, Moscow, 1976).
[33] G.F. Carey, Computer Methods in Appl. Mechanics and Engineering 35 (1982) 1.
[34] L.S. Solov'ev, JETP 26 (1968) 400.
[35] L.M. Degtyarev, V.I. Pistunovich and V.D. Shafranov, Nucl. Fusion 20 (1980) 102.
[36] B.B. Kadomtsev, in: Plasma Theory Questions, vol. 2, ed. M.A. Leontovich (Atomizdat, Moscow, 1963) p. 132.
[37] L.M. Degtyarev, V.V. Drozdov, A.A. Martynov and S.Yu. Medvedev, Intern. Conf. on Plasma Phys., Proc. invited papers, Lausanne (1984).
[38] L.M. Degtyarev and V.V. Drozdov, Preprint of the Keldysh Inst. of Appl. Math. AN USSR (1984) N 32.
[39] L.M. Degtyarev, V.V. Drozdov, M.I. Mikhajlov et al., Fizika Plazmy 11 (1985) 39.